Quinn finite total homotopy TQFT as a once-extended TQFT.

Topological Quantum Field Theory Seminar (University of Lisbon)

João Faria Martins (University of Leeds)

24th September 2021

LEVERHULME TRUST

UNIVERSITY OF LEEDS

Partially funded by the Leverhulme Trust research project grant:

RPG-2018-029: "Emergent Physics From Lattice Models of Higher Gauge Theory".

Joint work with Tim Porter.

Thanks are due to Fiona Torzewska, Alex Bullivant and Paul Martin.

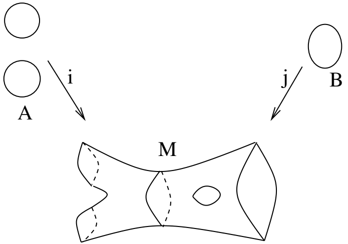
The category of manifold and cobordisms (sketch)

Let $n \in \mathbb{Z}_0^+$. Define symmmetric monoidal category $(\boldsymbol{n}, \boldsymbol{n}+1)$-**Cob**.

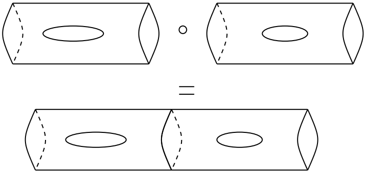
- Objects: closed smooth n-manifolds $A, B, \ldots$
- Morphisms $[M]\colon A \to B$ are equivalence classes of diagrams:

$$A \xrightarrow{\;i\;} M \xleftarrow{\;j\;} B\,.$$

Here M is a smooth $(n+1)$-manifold, and
i and j induce a diffeomorphism $\langle i, j\rangle\colon A \sqcup B \to \partial(M)$.



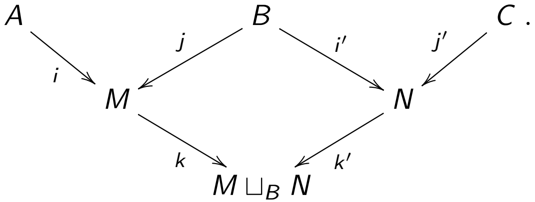
Visualization.



Composition of morphisms.
Note: Collars are required to construct smooth structure.

The category of manifold and cobordisms (sketch)

More precisely, the composition of cobordisms is via pushouts:



So $([M]\colon A \to B) \bullet ([N]\colon B \to C) = ([M \sqcup_B N]\colon A \to C)$.

Note that smooth structure on $M \sqcup_B N$ is not uniquely defined.
But it is unique up to diffeomorphism.

The monoidal structure in $(\boldsymbol{n}, \boldsymbol{n}+1)$-$\mathbf{Cob}$ is induced from the disjoint union of manifolds / cobordisms.

Topological quantum field theories

Definition (TQFT)

Given a non-negative integer $n \in \mathbb{Z}_0^+$,
a *Topological Quantum Field Theory (TQFT)*
is a symmetric monoidal functor:

$$\mathcal{F}\colon (\boldsymbol{n}, \boldsymbol{n}+1)\text{-}\mathbf{Cob} \to \mathbf{Vect}.$$

Plan of the talk

In this talk I will:

1. Recall Quinn's finite total homotopy TQFT
   $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}} \colon (\boldsymbol{n}, \boldsymbol{n}+1)\text{-}\mathbf{Cob} \to \mathbf{Vect}, \quad s \in \mathbb{C}$
   Where $\boldsymbol{\mathcal{R}}$ is a "homotopically finite space", a parameter.
   Cf. Frank Quinn. *Lectures on axiomatic topological quantum field theory.* In Geometry and quantum field theory. (1995)

2. Explain the combinatorial calculation of $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}$.
   For $\boldsymbol{\mathcal{R}}$ classifying space of a homotopy finite strict ω-groupoid (represented by a crossed complex of groupoids).

3. Explain the construction of a once-extended version of $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}$.

Homotopy finite spaces (or simplicial sets)

Definition (Homotopy finite space)

A space X (or simplicial set X) is called homotopy finite (HF) if:

- X has only a finite number of path components.
- Given a path-component K of X, exists $n \in \mathbb{N}$ such that:
  - $\pi_i(K)$ is trivial, if $i > n$.
  - $\pi_i(K)$ finite, if $i = 1, \ldots, n$.

Equivalently, X has finitely many path-components, and finitely many non-trivial homotopy groups, all of which are finite.

Definition (n-type)

Let $n \in \mathbb{N}$. A path-connected space X is called an n-type if:

1. X is homeomorphic to a CW-complex.
2. $\pi_i(X) = 0$, if $i > n$.

An n-type is HF if all of its homotopy groups are finite.

Classifying spaces of groups, etc

Example

Let G be a finite group. Classifying space B_G is path-connected. Also:

- $\pi_1(B_G, *) \cong G$, and
- $\pi_i(B_G, *) = 0, \text{ if } i \geq 2.$

So B_G is a finite 1-type. So B_G is a HF space.

More generally, if G is a finite groupoid, or finite 2-group, then classifying space B_G is homotopy finite

More examples later.

The homotopy content of a homotopy finite space

Definition (Homotopy content)

If X is HF, the *homotopy content* of X is:

$$\chi^{\pi}(X) = \sum_{K \in \pi_0(X)} \frac{|\pi_2(K)|\ |\pi_4(K)|\ |\pi_6(K)| \dots}{|\pi_1(K)|\ |\pi_3(K)|\ |\pi_5(K)| \dots} \in \mathbb{Q}.$$

Here $\pi_0(X)$ is the set of path-components of X.

Example (Classifying spaces of finite groups)

If G is a finite group then $\chi^{\pi}(B_G) = 1/|G|$.

The homotopy content first appeared (I think) in:
Frank Quinn. *Lectures on axiomatic topological quantum field theory*. In Geometry and quantum field theory. (1995)

John C. Baez and James Dolan. *From finite sets to Feynman diagrams.* In Mathematics unlimited-2001 and beyond (2001)

Some properties HF spaces and their homotopy content

- If X and Y are HF, then so are $X \times Y$ and $X \sqcup Y$, and:

$$\chi^\pi(X \times Y) = \chi^\pi(X) \times \chi^\pi(Y),$$

$$\chi^\pi(X \sqcup Y) = \chi^\pi(X) + \chi^\pi(Y).$$

- Let $p\colon E \to B$ be a fibration of HF spaces. Let $b \in B$. The fibre $F_b := p^{-1}(b)$ is HF.

  Moreover if B is path-connected then:

$$\chi^\pi(E) = \chi^\pi(B) \times \chi^\pi(F_b).$$

Cf. John C. Baez and James Dolan. *From finite sets to Feynman diagrams.* In Mathematics unlimited-2001 and beyond (2001)

Imma Gálvez-Carrillo, Joachim Kock, Andrew Tonks: *Homotopy linear algebra.* P ROY SOC EDINB A. (2018).

Function spaces and homotopy finite spaces

Theorem (Quinn)

Let M be a compact CW-complex. Let $\mathcal{R}$ is HF space. Then the function space below is HF:

$$\mathrm{TOP}(M, \mathcal{R}) = \{f\colon M \to \mathcal{R} \mid f \textit{ is continuous}\}.$$

In particular if M is a compact smooth manifold.

Note that $\mathrm{TOP}(M, \mathcal{R})$ is given the k-ification of the compact-open topology on the space of maps $M \to \mathcal{R}$.

Topology on $\mathrm{TOP}(M, \mathcal{R})$ is given by the sup distance if $\mathcal{R}$ is a locally-finite CW-complex (thus $\mathcal{R}$ is a metric space).

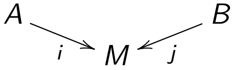
Quinn's (finite total homotopy) TQFT

Let $\boldsymbol{\mathcal{R}}$ be a HF-space. Let $s \in \mathbb{C}$.

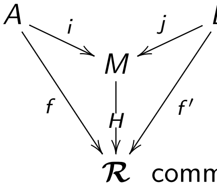
Functor: $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}} \colon (\boldsymbol{n}, \boldsymbol{n}+1)\text{-}\mathbf{Cob} \to \mathbf{Vect}$.

- If A is an n-manifold then:

$$\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}(A) = \mathbb{C}\big(\pi_0(\mathrm{TOP}(A, \boldsymbol{\mathcal{R}}))\big).$$

- Matrix elements assigned to cobordisms 

$$\Big\langle [f] \Big| \mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}(M) \Big| [f'] \Big\rangle := \chi^{\pi}\Big\{ H \colon M \to \boldsymbol{\mathcal{R}} :$$



$\Big\}$ commutes

$$\times \big(\chi^{\pi}(\mathrm{PC}_f(\mathrm{TOP}(A, \boldsymbol{\mathcal{R}})))\big)^{s} \; \big(\chi^{\pi}(\mathrm{PC}_{f'}(\mathrm{TOP}(B, \boldsymbol{\mathcal{R}})))\big)^{1-s}.$$

Here $\mathrm{PC}_x(X)$ is the path component of x in space X.

Discussion: Quinn finite total homotopy TQFT

Note: Quinn TQFT $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}$ can be twisted by classes in $H^{n+1}(\boldsymbol{\mathcal{R}}, U(1))$.

- Let G be a finite group. Let $\boldsymbol{\mathcal{R}}$ be classifying space of G. Then $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}$ coincides with Dijkgraaf-Witten TQFT.

  Explicitly calculable. Related to topological gauge theory.
  Related to Kitaev quantum double model.

- Let $\mathcal{G}$ be a finite 2 group. Let $\boldsymbol{\mathcal{R}}$ be classifying space of $\mathcal{G}$. $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}$ coincides with (twisted) Yetter TQFT (___ / Porter).

  Explicitly calculable. Related to topological higher gauge theory.
  Related to higher Kitaev models formulated with 2-groups.

This generalises.

Discussion: Quinn finite total homotopy TQFT

Theorem (Ellis)

Any connected HF space is homotopic to a space of the form $|\overline{W}(G)|$, where G is a finite simplicial group.

Graham Ellis: *Spaces with finitely many non-trivial homotopy groups all of which are finite.* Topology (1997)

Let $\boldsymbol{\mathcal{R}} = |\overline{W}(G)|$, and $M\colon A \to B$ a triangulated cobordism.
Can compute $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}(M\colon A \to B)$ using simplicial homotopy tools.

Quinn TQFT $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}\colon (\boldsymbol{n}, \boldsymbol{n}+1)\text{-}\mathbf{Cob} \to \mathbf{Vect}$ thus is combinatorial.
We will see the case when $\boldsymbol{\mathcal{R}}$ is the classifying space of a strict omega-groupoid.

This includes the case of classifying spaces of 2-groups, relevant for higher gauge theory.

Crossed modules of groups (as models for 2-types)

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
- A left action $\triangleright$ of G on E, by automorphisms,

  such that the following conditions *(Peiffer equations)* hold:

Peiffer 1 $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$

Peiffer 2 $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

E a group, $G = \mathrm{Aut}(E)$.

We have a crossed module $\left(E \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(E), \mathrm{eval}\right)$.

Theorem (Spencer)

Category of crossed modules is equivalent to category of 2-groups.

Theorem (Whitehead-MacLane)

Homotopy category of crossed modules is equivalent to homotopy category of 2-types. (Spaces X with $\pi_i(X) = 0$ if $i \geq 3$.)

Crossed complexes (of groupoids)

A crossed complex is given by a complex of groupoids over set C_0

$$\mathcal{C} := \ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

(hence all groupoids have object set C_0). Such that:

- All groupoids for C_i, $i \geq 2$ are totally disconnected.
- All boundary maps are the identity over the object C_0.
- We have an action of C_1 on all groupoids C_i, $i \geq 2$
- All boundary maps preserve actions.
- Peiffer 1: If $x \xrightarrow{g} y \in C_1$ and $K \in C_n(y,y)$ then (for $k \geq 2$):
$$\partial(g \triangleright K) = g\partial(K)g^{-1}$$
- Peiffer 2: If $K, L \in C_2(y,y)$ then $\partial(K) \triangleright L = KLK^{-1}$
- The action of $\partial(C_2)$ is trivial on all groupoids C_i for $i \geq 3$.
- C_i is abelian if $i \geq 3$.

Monoidal closed category Crs

The category Crs of crossed complexes is equivalent to the category of strict omega-groupoids (Brown–Higgins).

Also: Crs is a monoidal closed category (Brown–Higgins).

- Given $\mathcal{A}$ and $\mathcal{B}$ we can form tensor product $\mathcal{A} \otimes \mathcal{B}$.
- Given $\mathcal{A}$ and $\mathcal{B}$ we can form “function space” $\mathrm{CRS}(\mathcal{A}, \mathcal{B})$.
- Natural equivalence $\mathrm{Crs}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{Crs}(\mathcal{A}, \mathrm{CRS}(\mathcal{B}, \mathcal{C}))$.
- Given $\mathcal{A}$ and $\mathcal{B}$, $\pi_1(\mathrm{CRS}(\mathcal{A}, \mathcal{B}))$ is groupoid of maps $\mathcal{A} \to \mathcal{B}$ and (2-fold homotopy classes of) homotopies between them.

Example

Let G and H be finite groups, seen as a crossed complexes

- $G \otimes H$ is the free product $G * H$.
- $\mathrm{CRS}(G, H)$ is the groupoid with
  - objects maps $f\colon G \to H$.
  - morphisms $f \xrightarrow{h} f'$ are elements of H conjugating f into f'.

Fundamental crossed complexes of CW-complexes

Theorem (Brown-Higgins)

Let X be a CW-complex. Then the sequence of groupoids

$$\begin{aligned}\Pi(X) := \dots \xrightarrow{\partial} \pi_n(X^n, X^{n-1}, X^0) \xrightarrow{\partial} \pi_{n-1}(X^{n-1}, X^{n-2}, X^0) \\ \xrightarrow{\partial} \dots\dots \xrightarrow{\partial} \pi_2(X^2, X^1, X^0) \xrightarrow{\partial} \pi_1(X^1, X^0) \overset{s}{\underset{t}{\rightrightarrows}} X_0.\end{aligned}$$

is a totally free crossed complex with object set X_0.

Classifying spaces of Crossed complexes

Definition (Nerve and classifying space of crossed complexes)

The nerve $\mathcal{NC}$ of the crossed complex

$$\mathcal{C} = \ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

is the simplicial set $\mathcal{NC}$ such that

$$(\mathcal{NC})_n = \hom_{\mathrm{Crs}}\big(\Pi(\Delta(n)), \mathcal{C}\big).$$

The classifying space of $\mathcal{C}$ is $B_{\mathcal{C}} := |\mathcal{NC}|$.

Theorem (Brown-Higgins)

The homotopy groups of $B_{\mathcal{C}}$ are the homology groups of $\mathcal{C}$.

So if $\mathcal{C}$ is finite then $B_{\mathcal{C}}$ is a HF space.

$\mathcal{C}$-colourings ($\mathcal{C}$ a crossed complex)

Let M be a manifold with triangulation t. Consider:

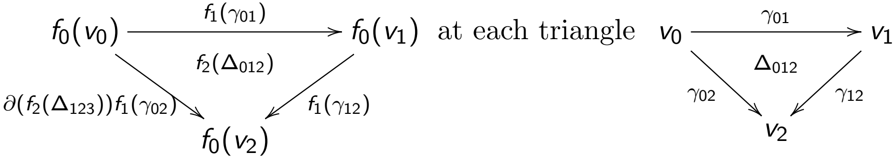
$$\mathcal{C} = \ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1 \underset{t}{\overset{s}{\rightrightarrows}} C_0$$

Maps $f\colon \Pi(M_t) \to \mathcal{C}$ are in 1-to-1 correspondence with $\mathcal{C}$-colorings:

- a map $f_0\colon \text{Vertices}(M_t) \to C_0$
- a map $f_1\colon \text{edges}(M_t) \to C_1$, looking like:

$$f_0(v_0) \xrightarrow{f_1(\gamma)} f_0(v_1) \text{ at each edge } v_0 \xrightarrow{\gamma} v_1 \text{ of } M_t$$

- a map $f_2\colon \text{triangles}(M_t) \to C_2$, looking like:



Rule: Colour of an n-face must fit with colour of its boundary.

Calculation of Quinn's $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}$ for $\boldsymbol{\mathcal{R}} = |\mathcal{N}(\mathcal{C})|$.

Let $\mathcal{C}$ be a pointed homotopically finite crossed complex
Hence $\boldsymbol{\mathcal{R}} := |\mathcal{N}\mathcal{C}|$ is a homotopically finite space.

Theorem (___ /Porter (following Brown-Higgins))

Let A be a closed n-manifold with a triangulation t. Then

$$\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}(A) \cong \mathbb{C}(\pi_0(\mathrm{CRS}(\Pi(A_t), \mathcal{C})).$$

Here $\mathrm{CRS}(\_, \_)$ is internal-hom in the cat. of crossed complexes.

In particular a basis of $\mathcal{F}^{(s)}_{\boldsymbol{\mathcal{R}}}(A)$ consists of homotopy classes of crossed complex maps $f : \Pi(A_t) \to \mathcal{C}$,
considered up to homotopy / pseudonatural equivalence.

Note $\mathrm{CRS}(\Pi(A_t), \mathcal{C})$ is the crossed complexes of all maps $f : \Pi(A_t) \to \mathcal{C}$ and their homotopies / natural transformations of all orders.

Calculation of Quinn $\mathcal{F}^{(s)}_{\mathcal{R}}$ for $\mathcal{R} = |\mathcal{N}(\mathcal{C})|$.

Consider a cobordism $A \xrightarrow{i} M \xleftarrow{j} B$

Consider a triangulation t of of M extending triangulations t of A and B.

Theorem (___/Porter (following Brown-Higgins))

Given $f\colon \Pi(A_t) \to \mathcal{C}$ *and* $f'\colon \Pi(A_t) \to \mathcal{C}$

$$\Big\langle [f] \Big| \mathcal{F}^{(s)}_{\mathcal{R}}(M) \Big| [f'] \Big\rangle = \#\Big\{ H\colon \Pi(M_t) \to \mathcal{C} : \begin{array}{ccccc} \Pi(A_t) & \xrightarrow{i} & \Pi(M_t) & \xleftarrow{j} & \Pi(B_t) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle f'} & \\ & & \mathcal{C} & & \end{array} \text{ commutes} \Big\}$$

$\times$*factors depending only on number of simplices of* A_t, B_t, M_t; *and* $\mathcal{C}$.

Extended cobordisms

Let $(\boldsymbol{n}, \boldsymbol{n}+1, \boldsymbol{n}+2)$-$\mathbf{Cob}$ be the bicategory with:

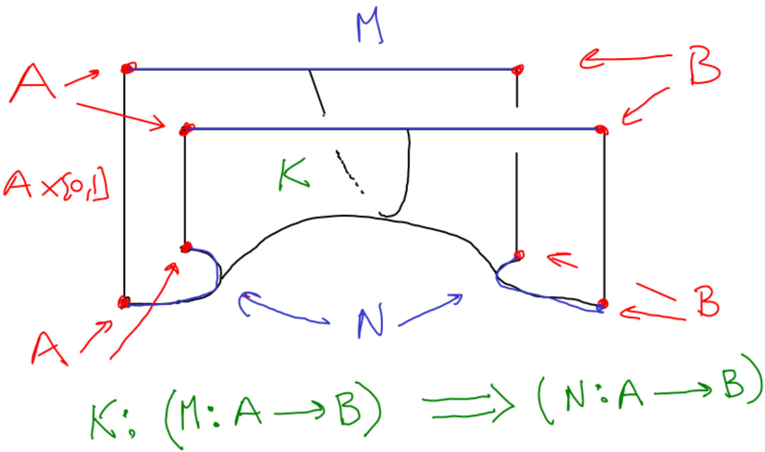
- Objects n-dimensional closed smooth manifolds $A, B, \ldots$
- 1-morphisms $M \colon A \to B$ are $(n, n+1)$-cobordisms (no equivalence relation is applied now):

$$A \xrightarrow{\ i\ } M \xleftarrow{\ j\ } B\ .$$

- 2-morphisms $K \colon M \implies N$ are $(n, n+1, n+2)$-extended-cobordisms (up to equivalence):

$$\begin{array}{ccccc}
A & \xrightarrow{\ i_1\ } & M & \xleftarrow{\ j_1\ } & B \\
{\scriptstyle \iota_0^A}\downarrow & & \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle \iota_0^B} \\
A \times [0,1] & \xrightarrow{\ i_E\ } & K & \xleftarrow{\ i_W\ } & B \times [0,1] \\
{\scriptstyle \iota_1^A}\uparrow & & \uparrow{\scriptstyle i_S} & & \uparrow{\scriptstyle \iota_1^B} \\
A & \xrightarrow[\ i_2\ ]{} & N & \xleftarrow[\ j_2\ ]{} & B
\end{array}$$

Extended cobordisms



Horizontal and vertical composition is performed via pushouts.
(As before, we need collars to construct smooth structures.)

(Once)-Entended TQFTs

A once-extended TQFT is a symmetric monoidal bifunctor:

$$\mathcal{F}\colon (\boldsymbol{n}, \boldsymbol{n}+1, \boldsymbol{n}+2)\text{-}\mathbf{Cob} \to \mathbf{Alg}.$$

Here $\mathbf{Alg}$ is some "algebraic" symmetric monoidal bicategory.

In this talk we will take:

- $\mathbf{Alg}$ to be the bicategory $\mathbf{Mor}$ with:
  - objects algebras $\mathcal{A}, \mathcal{B}$,...
  - with 1-morphisms $\mathcal{M}\colon \mathcal{A} \to \mathcal{B}$ being $(\mathcal{A}, \mathcal{B})$-bimodules $\mathcal{M}$.
  - Composition $\mathcal{A} \xrightarrow{\mathcal{M}} \mathcal{B} \xrightarrow{\mathcal{N}} \mathcal{C}$ is $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}$.
  - 2-morphisms $(\mathcal{A} \xrightarrow{\mathcal{M}} \mathcal{B}) \implies (\mathcal{A} \xrightarrow{\mathcal{M}'} \mathcal{B})$ are bimodule maps.
- We will also consider $\mathbf{Alg}$ to be the bicategory $\mathbf{Prof}$ with
  - objects homotopy finite groupoids $\mathcal{G}$, $\mathcal{H}$,
  - 1-morphisms $\mathcal{G} \to \mathcal{H}$ are (enriched) profunctors, i.e. functors $\mathcal{G}^{\mathrm{op}} \times \mathcal{H} \to \mathbf{Vect}$. Composition is via coends.
  - 2-Morphisms are natural transformations of functors.

Once-extended version of Quinn TQFT

Let $\mathcal{R}$ be a HF space. We have a once-extended TQFT

$$\widehat{\mathcal{Q}}_{\mathcal{R}} \colon (\boldsymbol{n}, \boldsymbol{n}+1, \boldsymbol{n}+2)\text{-}\mathbf{Cob} \to \mathbf{Prof}.$$

- If A is an n manifold then

$$\widehat{\mathcal{Q}}_{\mathcal{R}}(A) = \pi_1(\mathrm{TOP}(A, \mathcal{R}), \mathrm{TOP}(A, \mathcal{R})).$$

- A cobordism:

$$A \xrightarrow{i} M \xleftarrow{j} B\,.$$

  is sent to the profunctor:

$$\widehat{\mathcal{Q}}_{\mathcal{R}}(M) \colon \widehat{\mathcal{Q}}_{\mathcal{R}}(A)^{\mathrm{op}} \times \widehat{\mathcal{Q}}_{\mathcal{R}}(B) \to \mathbf{Vect}$$

  obtained from the path-space fibration:

$$\langle i^*, j^* \rangle \colon \mathrm{TOP}(M, \mathcal{R}) \to \mathrm{TOP}(A, \mathcal{R}) \times \mathrm{TOP}(B, \mathcal{R}).$$

  As per Quinn TQFT from here.

Decorated manifolds

Let $\mathcal{R}$ be a homotopy finite space.

Definition (Decorated manifold)

A decorated manifold $\mathsf{A} = (A, \overline{x}_A)$ is a manifold A together with a finite subset $\overline{x}_A$ of $\mathrm{TOP}(A, \mathcal{R})$, containing at least one element for each path component of $\mathrm{TOP}(A, \mathcal{R})$.

We have bicategory $(\boldsymbol{n}, \boldsymbol{n}+1, \boldsymbol{n}+2)$-$\overline{\mathbf{Cob}}$ of decorated manifolds, (undecorated) cobordisms and (undecorated) extended cobordism.

Finitary once-extended version of Quinn TQFT

Let $\boldsymbol{\mathcal{R}}$ be a homotopy finite space, n a non-negative integer.;

Theorem (Finitary extended Quinn TQFT)

We have bifunctor:

$$\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}} \colon (\boldsymbol{n}, \boldsymbol{n}+1, \boldsymbol{n}+2)\text{-}\overline{\mathbf{Cob}} \to \mathbf{Prof},$$

sending $\mathsf{A} = (A, \overline{x}_A)$ *to* $\pi_1(\mathrm{TOP}(A, \boldsymbol{\mathcal{R}}), \overline{x}_A)$.

Note 1: The groupoid $\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}(\mathsf{A}, \overline{x}_A)$ is finite.
Previously the groupoid: $\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}(A) = \pi_1(\mathrm{TOP}(A, \boldsymbol{\mathcal{R}}), \mathrm{TOP}(A, \boldsymbol{\mathcal{R}}))$
had an uncountable number of objects.

Note 2: Let A be an n-manifold.
If $\overline{x}_A$ and $\overline{y}_A$ are different decorations of A then

$$\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}\big((A, \overline{x}_A) \xrightarrow{A \times I} (A, \overline{y}_A)\big).$$

gives a canonical profunctor $\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}\big((A, \overline{x}_A)\big) \to \overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}\big((A, \overline{y}_A)\big)$.

Morita valued extended version of Quinn TQFT

Let $\boldsymbol{\mathcal{R}}$ be a homotopy finite space and $n \in \mathbb{Z}_0^+$.

Theorem (Morita valued once-extended Quinn TQFT)

The bifunctor:

$$\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}} \colon (\boldsymbol{n}, \boldsymbol{n}+\mathbf{1}, \boldsymbol{n}+\mathbf{2})\text{-}\overline{\mathbf{Cob}} \to \mathbf{Prof},$$

sending $\mathsf{A} = (A, \overline{x}_A)$ *to* $\pi_1(\mathrm{TOP}(A, \boldsymbol{\mathcal{R}}), \overline{x}_A)$,

"linearises" to a bifunctor, denoted:

$$\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\boldsymbol{\mathcal{R}}} \colon (\boldsymbol{n}, \boldsymbol{n}+\mathbf{1}, \boldsymbol{n}+\mathbf{2})\text{-}\overline{\mathbf{Cob}} \to \mathbf{Mor},$$

sending $\mathsf{A} = (A, \overline{x}_A)$ *to groupoid algebra* $\mathbb{C}(\pi_1(\mathrm{TOP}(A, \boldsymbol{\mathcal{R}}), \overline{x}_A))$.

The case of crossed complexes / strict omega-groupoids

Suppose that $\mathcal{R} = B_{\mathcal{C}}$, where $\mathcal{C}$ is a finite crossed complex.

Theorem

If A has a triangulation A_t then A is naturally decorated. Moreover:

$$\overline{\mathcal{Q}}_{\mathcal{R}}(A_t) = \pi_1\big(\mathrm{CRS}(\Pi(A_t), \mathcal{C})\big),$$

$$\overline{\mathcal{Q}}_{\mathcal{R}}^{\mathbf{Mor}}(A_t) = \mathbb{C}\big(\pi_1(\mathrm{CRS}(\Pi(A_t), \mathcal{C}))\big).$$

Note: $\pi_1\big(\mathrm{CRS}(\Pi(A_t), \mathcal{C})\big)$ *is groupoid of crossed complex maps $\Pi(A_t) \to \mathcal{C}$, and (2-fold homotopy classes of) homotopies between them.*

This permits the computation of once-extended Quinn TQFT $\overline{\mathcal{Q}}_{\mathcal{R}}$ and $\overline{\mathcal{Q}}_{\mathcal{R}}^{\mathbf{Mor}}$ if $\mathcal{R}$ is the classifying space of a finite crossed complex.

Computations $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\boldsymbol{\mathcal{R}}}, \overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}} \colon (\boldsymbol{n}, \boldsymbol{n}+1, \boldsymbol{n}+2)\text{-}\overline{\mathbf{Cob}} \to \mathbf{Alg}$

Consider n-manifolds A to come with a (singular) triangulation, a homeomorphism $|S| \to A$ where S is a simplicial set.

This gives rise to once-extended TQFTs:
$\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\boldsymbol{\mathcal{R}}}, \overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}} \colon (\boldsymbol{n}, \boldsymbol{n}+1, \boldsymbol{n}+2)\text{-}\mathbf{Cob} \to \mathbf{Alg}$

Example

Let G be a finite group. $\boldsymbol{\mathcal{R}} = B_G$. Let A be a pointed n-manifold.

$$\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}(A) = \hom(\pi_1(A), G)//G$$

Action groupoid of action of G on set of morphisms $\pi_1(A) \to G$.

= groupoid of G-connections on A and their gauge transformations

Cf. Jeffrey Morton. *Cohomological twisting of 2-linearization and extended TQFT.* J. Homotopy Relat. Struct. (2015).

Computations $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\mathcal{R}}, \overline{\mathcal{Q}}_{\mathcal{R}} \colon (n, n+1, n+2)\text{-}\overline{\mathbf{Cob}} \to \mathbf{Alg}$

Let G be a finite group. $\mathcal{R} = B_G$.

- $n = 0$, then $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\mathcal{R}}(.) = \mathbb{C}(G)$
- $n = 1$, then $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\mathcal{R}}(S^1)$ is the quantum double of $\mathbb{C}(G)$.
- $n = 2$, then $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\mathcal{R}}(S^2) = \mathbb{C}$.
- $n = 2$, then $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\mathcal{R}}(S^1 \times S^1)$ is the groupoid algebra of the action groupoid of G acting on $\{(a, b) \in G \times G \mid ab = ba\}$ by simultaneous conjugation.
- ... All easy to compute.

In particular this gives new proof that there exists a (1,2,3)-extended TQFT sending S^1 to the quantum double of G.

Computations $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\mathcal{R}}, \overline{\mathcal{Q}}_{\mathcal{R}} \colon (n, n+1, n+2)\text{-}\overline{\mathbf{Cob}} \to \mathbf{Alg}$

Example

Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a finite crossed module. $\mathcal{R} = B_{\mathcal{G}}$.

$\overline{\mathcal{Q}}_{\mathcal{R}}(S^1)$ is the groupoid with objects: $g \in G$.
Morphisms in $\overline{\mathcal{Q}}_{\mathcal{R}}(S^1)$ are equivalence classes of arrows like:

$$g \xrightarrow{[(h,e)]} \partial(e)hgh^{-1}, \quad g, h \in G, \quad e \in E.$$

$=$ groupoid of $\mathcal{G}$-2-connections on S^1, with morphisms equivalence classes of gauge transformation (up to 2-gauge transformations)
Latter is always the case.

Cf. Alex Bullivant and Clement Delcamp. *Tube algebras, excitations statistics and compactification in gauge models of topological phases.* JHEP (2019)
Alex Bullivant and Clement Delcamp. *Excitations in strict 2-group higher gauge models of topological phases.* JHEP (2020).

Thanks!

Main References:

João Faria Martins and Tim Porter:

• *On Yetter's invariant and an extension of the Dijkgraaf-Witten invariant to categorical groups.* Theory Appl. Categories (2007)

• *A rendition and an extension of Quinn finite total homotopy TQFT: Applications to once-extended TQFTs derived from finite 2-groups and finite strict omega-groupois.* Soon appearing.

Some technical to results on the text.

- Brown R., Higgins P.J.: Colimit Theorems for Relative Homotopy Groups, J. Pure Appl. Algebra 22 (1981).
- *R. Brown, P. Higgins, R Sivera*: Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids.European Mathematical Society (2011)
- *H. J. Baues*: Combinatorial homotopy and 4-dimensional complexes. Berlin etc.: Walter de Gruyter (1991)