Discrete Hamitonian models for 3+1 D topological phases derived from higher gauge theory

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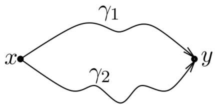
(Discrete) gauge theory and holonomy

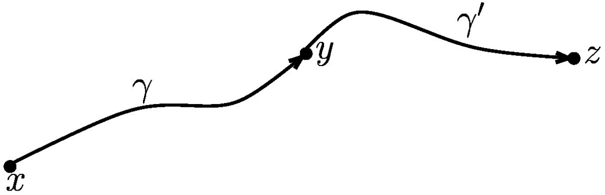
- Let M be a manifold.
- A path in M is a piecewise smooth map $\gamma\colon [0,1] \to M$.
  We consider paths up to homotopy, relative to the end-points.



Paths γ_1 and γ_2 are homotopic.

- Denote paths as $(x \xrightarrow{\gamma} y)$, x and y are initial and end-points.
- Paths $(x \xrightarrow{\gamma} y)$ and $(y \xrightarrow{\gamma'} z)$ can be concatenated:

$$(x \xrightarrow{\gamma} y)(y \xrightarrow{\gamma'} z) = (x \xrightarrow{\gamma\,\gamma'} z).$$



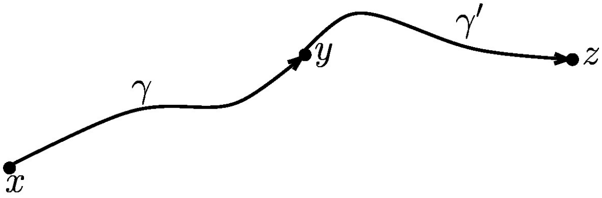
Gauge Theory and Holonomy

Let G be a group (G will be finite throughout the talk).
Given a principal G-bundle $P \to M$ – i.e. a gauge field –,
we have the parallel transport (a.k.a. holonomy) of P:

$$\mathcal{F} \colon \{Paths \quad in \;\; M\} \to G$$
$$\gamma \longmapsto \mathrm{hol}^1(\gamma) = g_\gamma \in G$$

Recall parallel transport preserves concatenation of paths:

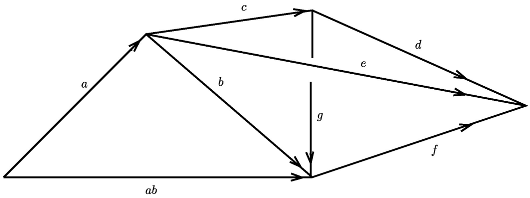
$$\mathcal{F}\big((x \xrightarrow{\gamma} y)(y \xrightarrow{\gamma'} z)\big) = \mathcal{F}(x \xrightarrow{\gamma} y)\;\mathcal{F}(y \xrightarrow{\gamma'} z)$$



NB: must specify elements $p_v \in F_v$, the fibre of P at each $v \in M$.
If G is a Lie group we need G-connection A. Locally $A \in \Omega^1(M, \mathfrak{g})$.

Gauge Theory and Holonomy

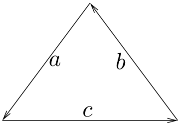
Conversely, G-connections can be defined from their holonomy.
Since G is finite, and M compact, to reconstruct the G-connection we only need to know the holonomy along a finite number of paths.
The theory of gauge fields becomes combinatorial / discrete.
Combinatorially, a G-connection over M looks like:



$a, b, c,$
$d, e, f, g \in G.$

Labels on edges denote holonomy along them.
Flatness conditions are satisfied on triangles: the holonomy around each triangle is trivial.

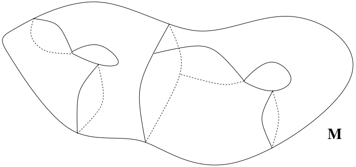
The holonomy around a more complicated polygon (plaquette) should also be trivial.

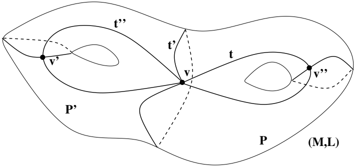


$abc = 1_G$

Discrete G-gauge fields: G a finite group

- Let M be a manifold with a CW-decomposition L into 'cells': vertices $v, v', v'' \in L^0$, edges $t, t', t'' \in L^1$, plaquettes $P, P' \in L^2$, blobs $b, b' \in L^3$, ...





(The interior of each plaquette P should be an open disk.)

- An edge $t = (v \xrightarrow{t} v'') \in L^1$ is assigned $g_t = \mathcal{F}(v \xrightarrow{t} v'') \in G$, the holonomy along t.
  Multiplicativity of holonomy lets us know holonomy along paths homotopic to paths obtained from concatenating edges.
- Each vertex $v \in L^0$ carries a copy of G (to be the group of gauge operators supported in v).
- Each plaquette $P \in L^2$ imposes a flatness condition on the colours of the edges around P.

Kitaev Quantum Double Model for topological phases

Define: a) Hilbert space $V(M,L) = \mathbb{C}\{Functions\ \mathcal{F}\colon L^1 \to G\}$.
(One copy of G for each edge $t \in L^1$.)

b) a group $T(M,L) = \prod_{v \in L^0} G$ of gauge operators $U\colon L^0 \to G$.
(One copy of G for each vertex $v \in L^0$.)

Given $v \in L^0$, $g \in G$, put $U_v^g \in T(M,L)$ to be the gauge operator (called vertex operator) such that:

$$U_v^g(x) = \begin{cases} g, \text{ if } x = v \\ 1_G, \text{ otherwise} \end{cases}$$

Left-action of $T(M,L)$ on $V(M,L)$, by gauge transformations:
Let $\mathcal{F} \in V(M,L)$ and $U \in T(M,L)$, define:

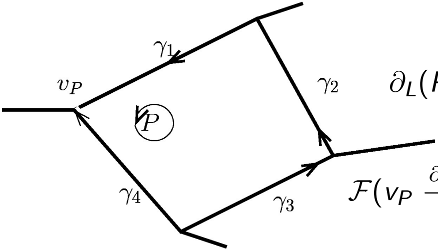
$$(U.\mathcal{F})(x \xrightarrow{t} y) = U(x)\mathcal{F}(x \xrightarrow{t} y)U(y)^{-1}.$$

So $$(U_v^g.\mathcal{F})(x \xrightarrow{t} y) = \begin{cases} g\,\mathcal{F}((x \xrightarrow{t} y)); v = x, v \neq y, \\ \mathcal{F}((x \xrightarrow{t} y))\,g^{-1}; v = y, v \neq x, \\ g\mathcal{F}((x \xrightarrow{t} y))\,g^{-1}; v = x = y. \end{cases}$$

$U_g^v\colon V(M,L) \to V(M,L)$ is unitary and called a vertex operator.

Plaquette operators

- Each plaquette P must be assigned a base-point v_P.
- A plaquette $P \in L^2$ attaches to M^1 (the union of all 1-cells) along a path in M^1, namely $\partial_L(P) = \left(v_P \xrightarrow{\partial_L(P)} v_P \right)$



$$\partial_L(P) = \left(v_P \xrightarrow{\gamma_4^{-1}\gamma_3\gamma_2\gamma_1} v_P \right)$$

$$\mathcal{F}\left(v_P \xrightarrow{\partial_L(P)} v_P \right) = g_{\gamma_4}^{-1} g_{\gamma_3} g_{\gamma_2} \gamma_1$$

- Given a plaquette P and $g \in G$, define the plaquette operator:

$$\mathcal{D}_P^g(\mathcal{F}) = \begin{cases} \mathcal{F}, & \text{if } \mathcal{F}\left(v_P \xrightarrow{\partial_L(P)} v_P \right) = g \\ 0, & \text{otherwise} \end{cases}$$

- Plaquette operator $\mathcal{D}_P^g \colon V(M, L) \to V(M, L)$ is self-adjoint.

The Kitaev Quantum Double Model (quant-ph/9707021)

(Slightly different language, as in 1702.00868 [math-ph])
M with CW-decomposition L. $V(M,L) = \mathbb{C}\{\mathcal{F}\colon L^1 \to G\}$.
Consider the Hamiltonian $H\colon V(M,L) \to V(L,M)$:

$$H = -\sum_{v \in L^0} \frac{1}{|G|} \sum_{g \in G} U_v^g - \sum_{P \in L^2} \mathcal{D}_P^{1_G} = -\sum_{v \in L^0} \mathcal{A}_v - \sum_{P \in L^2} \mathcal{D}_P^{1_G}$$

All the $\mathcal{A}_v$ and $\mathcal{D}_P^{1_G}$ are commuting, self-adjoint, projectors.
$\mathcal{A}_v$ imposes gauge-invariance at $v \in L^0$.
$\mathcal{D}_P^{1_G}$ imposes 'flatness' around the boundary of $P \in L^2$.
Topological excitations are modules over the algebra $\langle \{ U_v^g, \mathcal{D}_P^h \} \rangle$.
Theorem: The ground state $GS(M,L)$ of H is:

$$GS(M,L) \;=\; \left\{ \mathcal{F} \in V(M,L) \;\middle|\; \begin{array}{c} U_v^g \triangleright \mathcal{F} = \mathcal{F}, \text{ for all } g \in G, v \in L^0 \\ \mathcal{D}_P^{1_G} \mathcal{F} = \mathcal{F}, \text{ for all } P \in L^2 \end{array} \right\}$$

$GS(M,L) \cong \mathbb{C}\{Maps\ M \to B_G\}/homotopy$, canonically.
Here B_G is the classifying space of G.
Hence $GS(M,L) = V(M)$ does not depend on L and only on M.

‘Extension’ of Kitaev model to Higher Gauge Theory

- Higher gauge theory is a higher order version of gauge theory.
- Higher gauge theory formalises non-abelian holonomy along paths, and also non-abelian holonomy along surfaces.
- Non-abelian holonomy along surfaces is multiplicative with respect to the several ways we can concatenate surfaces.

  (This is why higher category theory arises here.)

- We need a higher order version of a group: called a “2-group”.
- 2-groups are equivalent to crossed modules.

  A crossed module of groups $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:
  - a group map $\partial\colon E \to G$,
  - and a left-action of G on E, by automorphisms, such that:

  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}$, if $g \in G$ and $e \in E$;
  2. $\partial(e) \triangleright e' = ee'e^{-1}$, if $e, e' \in E$.

Crossed modules will mostly be finite throughout the talk.

Examples of crossed modules of groups $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

1. G a group; A and abelian group.
   Consider a left-action $\triangleright$ of G on A, by automorphisms.
   We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \mapsto 1_G} G, \triangleright)$.

   In the general example above we can for instance put:
   - $G = \{\pm 1, \times\}$. $A = (\mathbb{Z}_3, +)$. $g \triangleright a = ga \ (mod\ 3)$.
   - $G = \mathrm{GL}(\mathbb{Z}_p, n)$; i.e. $n \times n$ invertible matrices in $\mathbb{Z}_p$.
     $A = (\mathbb{Z}_p)^n$. Here p is a prime.

2. Given a group H, put $\mathcal{G} = (H \xrightarrow{g \mapsto \mathrm{Ad}_g} \mathrm{Aut}(H))$.
   Here $\mathrm{Ad}_g(x) = gxg^{-1}$.
   $\mathrm{Aut}(H)$ is the automorphism group of H.

2-dimensional (i.e. surface) holonomy functors

Given $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ we can define "bigons" in $\mathcal{G}$.

$$\begin{tikzcd}
{} \arrow[r, bend left, "\partial(e)^{-1}g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow e"] & {}
\end{tikzcd}, \quad g \in G, e \in E.$$

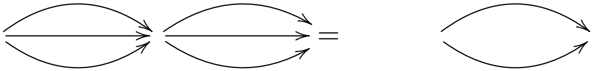
These compose horizontally and vertically:

$$\begin{tikzcd}
{} \arrow[r, bend left, "\partial(e)^{-1}g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Uparrow e"] & {} \arrow[r, bend left, "\partial(e')^{-1}h"] \arrow[r, bend right, "h"'] \arrow[r, phantom, "\Uparrow e'"] & {}
\end{tikzcd} = \begin{tikzcd}
{} \arrow[r, bend left, "\partial(e)^{-1}g\ \partial(e')^{-1}h"] \arrow[r, bend right, "gh"'] \arrow[r, phantom, "\Uparrow (g \triangleright e')e"] & {}
\end{tikzcd}$$

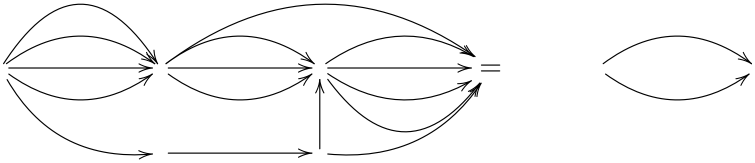
$$\begin{tikzcd}
{} \arrow[r, bend left=60, "\partial(e')^{-1}\partial(e)^{-1}g"] \arrow[r, "\partial(e)^{-1}g" description] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, bend left=30, "\Uparrow e'"] \arrow[r, phantom, bend right=30, "\Uparrow e"] & {}
\end{tikzcd} = \begin{tikzcd}
{} \arrow[r, bend left=60, "\partial(e')^{-1}\partial(e)^{-1}g"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\Uparrow (ee')"] & {}
\end{tikzcd}$$

2-dimensional holonomy functors

Horizontal and vertical compositions of bigons in $\mathcal{G}$ are:
associative, and have units and inverses.
The interchange law is satisfied. This means that the evaluation of



does not depend on the order whereby it is performed.
As a consequence evaluations of more complicated diagrams like:



do not depend on the order whereby we apply compositions.
A very general result is in 1702.00868 [math-ph]
This leads to a notion of non-abelian multiplication along surfaces.
This notion underpins surface-holonomy in higher gauge theory.

2-dimensional holonomy

A geometric bigon on in a manifold M is given by:
Two paths $\gamma, \gamma' \colon [0,1] \to M$, with the same initial and end-point.
A homotopy (i.e. a 'surface') $\Sigma \colon [0,1]^2 \to M$, connecting γ and γ'.
Σ is considered up to homotopy relative to $\partial([0,1]^2)$.
Geometric bigons are represented as:

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma'"] \arrow[r, bend right, "\gamma"'] \arrow[r, phantom, "\Uparrow \Sigma"] & y, \end{tikzcd}$$

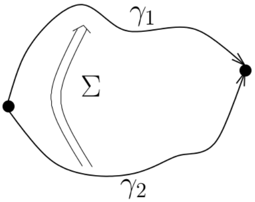
Geometric bigons can be concatenated horizontally and vertically.

- **Definition** Let M be a manifold; $\mathcal{G}$ a crossed module.
  A 2-dimensional holonomy (i.e. a higher gauge field) is a map:

$$\{\textit{Geometric bigons in } M\} \xrightarrow{\mathcal{F}} \{\textit{Bigons in } \mathcal{G}\}$$

Preserving horizontal and vertical compositions.
The underlying $\mathcal{G}$-2-bundle can be reconstructed from $\mathcal{F}$.

2D holonomy along Σ



$$\longmapsto \begin{tikzcd} \bullet \arrow[rr, bend left, "\mathrm{hol}^1(\gamma_1)"] \arrow[rr, bend right, "\mathrm{hol}^1(\gamma_2)"'] \arrow[rr, phantom, "\Uparrow \mathrm{hol}^2(\Sigma)"] & & \bullet \end{tikzcd} \quad = \quad \begin{tikzcd} {} \arrow[rr, bend left, "\partial(e_\Sigma)^{-1}g_{\gamma_2}"] \arrow[rr, bend right, "g_{\gamma_2}"'] \arrow[rr, phantom, "\Uparrow e_\Sigma"] & & {} \end{tikzcd},$$

Note: for Lie crossed modules $(\partial\colon E \to G, \triangleright)$, 2-dimensional holonomies arise from pairs $A \in \Omega^1(M, \mathfrak{g})$ and $B \in \Omega^2(M, \mathfrak{e})$, with $\partial(B) = Curv_A = dA + \frac{1}{2}[A, A]$.

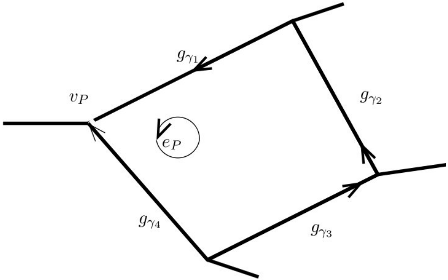
The HGT analogue of Kitaev quantum double model

Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a crossed module.
Let M be a compact manifold, possibly with boundary.
Let $L = (L^0, L^1, L^2, L^3 \dots)$ be a CW-decomposition of M.
In HGT 3-cells $b \in L^3$ (called blobs) have an important role.

A **discrete 2-connection** $\mathcal{F}$ is given by an assignment:

$$\gamma \in L^1 \mapsto g_\gamma \in G \text{ and } P \in L^2 \mapsto e_P \in E,$$

satisfying the **fake-flatness condition**, namely:
If we have a configuration like:



Then:

$$\partial(e_P) = g_{\gamma_4}^{-1} g_{\gamma_3} g_{\gamma_2} g_{\gamma_1}.$$

The Hilbert space for the higher Kitaev model

- M a compact manifold, with a CW-decomposition L.
- We put $\Phi(M,L) = \{Discrete\ 2 - connections\ \ \mathcal{F}\}$.
- And $V(M,L) = \mathbb{C}\Phi(M,L)$. Hilbert space for discrete HGT.
- The group of gauge operators puts together gauge transformations along vertices and along edges:

$$T(M,L) = \Big(\prod_{v \in L^0} G\Big) \ltimes \Big(\prod_{\sigma(t) \xrightarrow{t} \tau(t) \in L^1} E\Big)$$

$$= \{Functions\ L^0 \to G\} \ltimes \{Functions\ L^1 \to E\}$$

Where $U \in \prod_{v \in L^0} G$ left-acts in $\eta \in \prod_{t \in L^1} E$ as:

$$(U.\eta)(\sigma(t) \xrightarrow{t} \tau(t)) = U(\sigma(t)) \triangleright \eta((\sigma(t) \xrightarrow{t} \tau(t)))$$

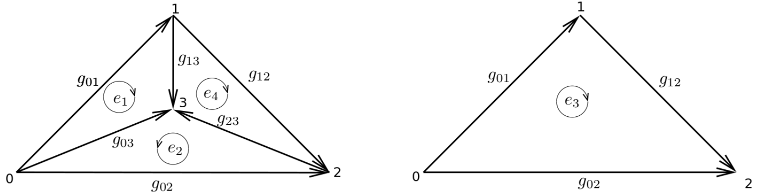
For S^1 with one vertex and one edge $T(S^1, L) = G \ltimes E$.

Discrete surface holonomy. arXiv:1702.00868

Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a crossed module.
Let $\mathcal{F} \in \Phi(M, L)$ be a discrete 2-connection.

- **Theorem** Let Σ be a 2-sphere cellularly embedded in M, $v \in \Sigma$, an 'initial point'. We have a surface-holonomy: $Hol^2_v(\mathcal{F}, \Sigma) \in \ker(\partial) \subset E$.
  This surface-holonomy depends only on the starting point $v \in \Sigma$, and not in the way whereby we combine 2-cells.

  For example, consider the discrete 2-connection on the tetrahedron Σ, below, based on the bottom left corner v_0.



$\partial_{\mathcal{G}}(e_1) = g_{01}g_{13}(g_{03})^{-1}$ $\quad$ $\partial_{\mathcal{G}}(e_4) = g_{12}g_{23}(g_{13})^{-1}$ $\quad$ $\partial_{\mathcal{G}}(e_2) = g_{02}g_{23}(g_{03})^{-1}$ $\quad$ $\partial_{\mathcal{G}}(e_3) = g_{01}g_{12}(g_{02})^{-1}$

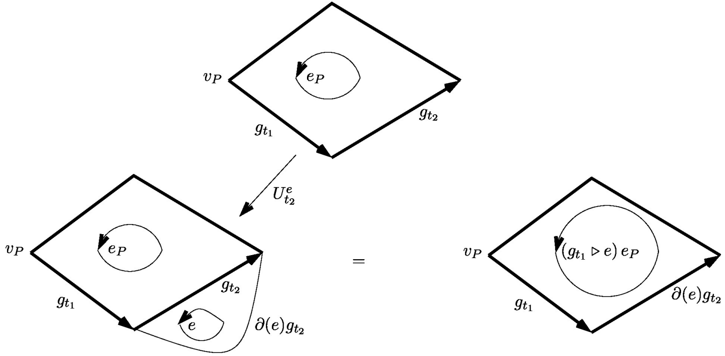
Then $\mathrm{Hol}^2_{v_0}(\mathcal{F}, \Sigma) = e_1 \ e_2^{-1} \ e_3^{-1} \ g_{01} \triangleright e_4$

Action of the group of gauge operators

- We have an action of the group of gauge operators $T(M, L)$ on $\Phi(M, L)$, preserving 2D holonomy, up to acting by $g \in G$.

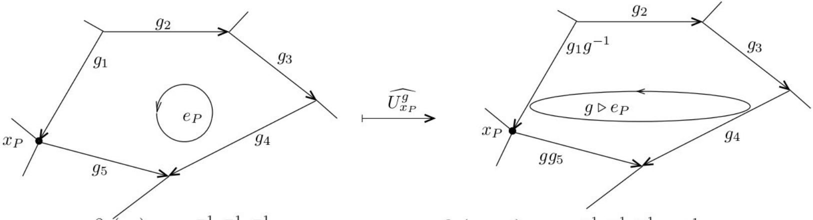
Given $t \in L^1$, and $e \in E$, let U_t^e be the unique gauge operator supported in t with $U_t^e(t) = e$. (Called an edge gauge spike.)

Given $v \in L^0$, and $g \in G$, let U_t^e be the unique gauge operator supported in v with $U_v^g(v) = g$. (Called a vertex gauge spike.)

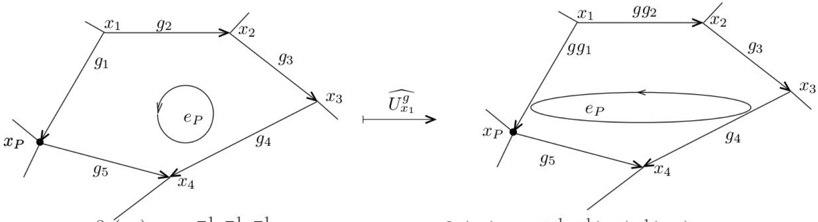


Action of the group of gauge operators

Some examples of vertex gauge transformations:



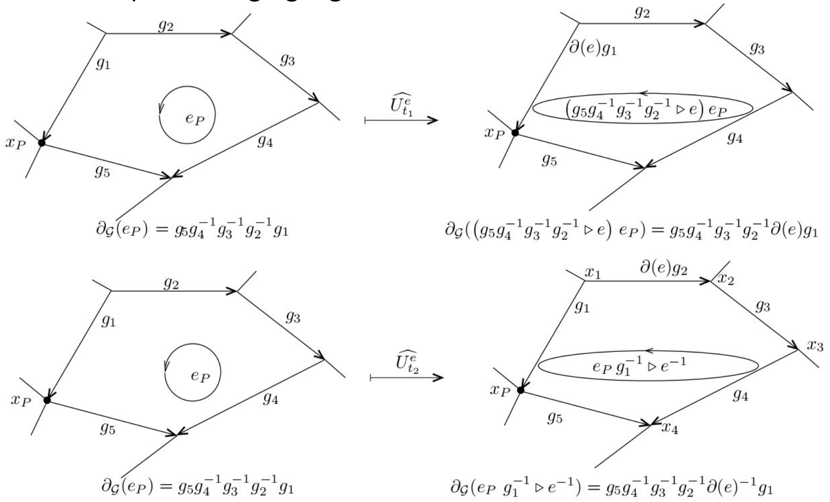
$\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1$ $\quad$ $\partial_{\mathcal{G}}(g \triangleright e_P) = g g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1 g^{-1}$



$\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} g_2^{-1} g_1$ $\quad$ $\partial_{\mathcal{G}}(e_P) = g_5 g_4^{-1} g_3^{-1} (gg_2)^{-1} (gg_1)$

Action of the group of gauge operators

Some examples of edge gauge transformations:



Vertex, edge and blob operators

Let $\mathcal{G} = (\partial\colon E \to G)$ be a crossed module.

- All vertex operators $U_v^g\colon V(M, L) \to V(M, L)$ are unitary.
- All edge operators $U_t^e\colon V(M, L) \to V(M, L)$ are unitary.
- Given a 3-cell b, a blob, let $\partial b \subset M$ be its boundary.
  Hence ∂b is a 2-sphere cellularly embedded in M.
  The blob operator $\mathcal{C}_b^k$ is defined as (here $k \in \ker(\partial)$)

$$\mathcal{C}_b^k(\mathcal{F}) = \begin{cases} \mathcal{F}, & \text{if } 2hol(\mathcal{F}, \partial b) = k \\ 0, & \text{otherwise} \end{cases} .$$

- Clearly $\mathcal{C}_b^k\colon V(M, L) \to V(M, L)$ is self-adjoint.

The higher gauge theory Kitaev model

$V(M,L) = \mathbb{C}\{Discrete\ 2-connections\ \ \mathcal{F}\}$.
Hamiltonian $H\colon V(M,L) \to V(M,L)$.

$$H = -\sum_{v \in L^0} \frac{1}{|G|} \sum_{g \in G} \hat{U_v^g} - \sum_{t \in L^1} \frac{1}{|E|} \sum_{e \in E} \hat{U_t^e} - \sum_{b \in L^3} \mathcal{C}_b^{1_E}.$$

$$H = -\sum_{v \in L^0} \mathcal{A}_v - \sum_{t \in L^1} \mathcal{B}_t - \sum_{b \in L^3} \mathcal{C}_b^{1_E}.$$

All operator in the last sum are commuting self-adjoint projectors.

$\mathcal{C}_b^{1_E}$ forces the surface-holonomy of a discrete 2-connection $\mathcal{F}$ to be trivial along the boundary of the 3-cell b.

Algebra generated by the U_t^g, U_t^e and C_b^k is our proposal for a local operator algebra $\mathcal{A}$. Relations are in arXiv:1702.00868.
Physically relevant (i.e. topological) excitations are module over $\mathcal{A}$.

Ground state degeneracy of higher Kitaev model

Theorem The ground state of $H\colon V(M,L) \to V(M,L)$ is

$$GS(M,L) = \left\{ \mathcal{F} \in V(M,L) \middle| \begin{array}{c} U_v^g \mathcal{F} = \mathcal{F}, \text{ for all } v \in L_0, g \in G \\ U_t^e \mathcal{F} = \mathcal{F}, \text{ for all } t \in L_1, e \in E \\ \mathcal{C}_b^{1_G} \mathcal{F} = \mathcal{F}, \text{ for all } b \in L_3 \end{array} \right\}$$

$$\cong \mathbb{C}\{Maps \;\; M \to B_{\mathcal{G}}\}/Homotopy, \text{ canonically } .$$

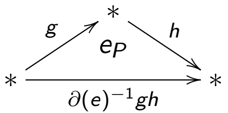
Hence $G(M,L) = V(L)$ depends only on M and not on L.

Here $B_{\mathcal{G}}$ is the classifying space of the crossed module $\mathcal{G}$.

Classifying space $B_{\mathcal{G}}$ of a crossed module $\mathcal{G}$

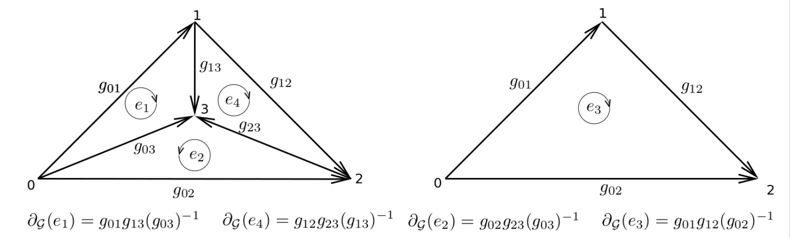
As the geometric realisation of a simplicial set $B_{\mathcal{G}}$ has:

- one 0-simplex $\{*\}$
- One 1-simplex $* \xrightarrow{g} *$ for each $g \in G$.
- 2-simplices have the form (where $g, h \in G$ and $e \in E$):



Plaquette P is based at bottom left vertex, and attaches clockwise.

- 3-simplices have the form (where $e_1 \ e_2^{-1} \ e_3^{-1} \ g_{01} \triangleright e_4 = 1_G$):



- The n-simplices are analogous. Colourings of 1 and 2-cells of the n-simplex, fake-flat on trianges and flat on tetrahedra,

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