Revisiting Quinn's Finite Total Homotopy TQFT.

Quantum Maths Seminar (University of Nottingham)

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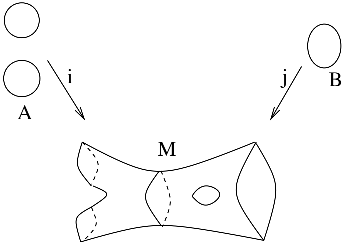
The category of manifold and cobordisms (sketch)

Consider the (symmmetric monoidal) category $\boldsymbol{(n, n+1)}$-$\mathbf{Cob}$.

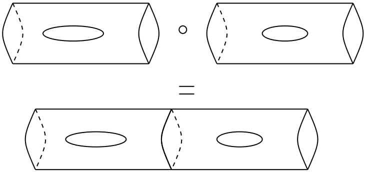
- Objects: closed n-manifolds $A, B, \ldots$
- Morphisms $[M]\colon A \to B$ are equivalence classes of diagrams:

$$A \xrightarrow{i} M \xleftarrow{j} B\,.$$

Here M is a smooth $(n+1)$-manifold, and
i and j induce a diffeomorphism $\langle i, j\rangle\colon A \sqcup B \to \partial(M)$.



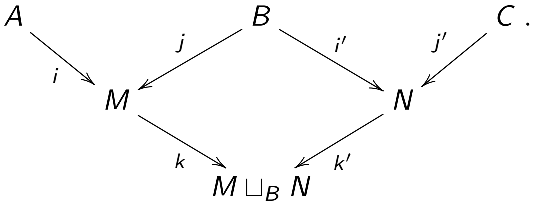
Visualization.



Composition of morphisms.
Collars are required to construct smooth structure.

The category of manifold and cobordisms (sketch)

More precisely, the composition of cobordisms is via pushouts:



So $([M]\colon A \to B) \bullet ([N]\colon B \to C) = ([M \sqcup_B N]\colon A \to C)$.

Topological quantum field theories

Definition (TQFT)

Given a non-negative integer n,
a *Topological Quantum Field Theory (TQFT)*
is a symmetric monoidal functor:

$$\mathcal{F}\colon \boldsymbol{(n, n+1)}\textbf{-Cob} \to \textbf{Vect}.$$

In this talk I will:

- Recall Quinn's finite total homotopy TQFT
  $\mathcal{F}^{(s)}_{\mathbb{B}}\colon \boldsymbol{(n, n+1)}\textbf{-Cob} \to \textbf{Vect}$,
  Here $\mathbb{B}$ is a "homotopically finite space": a parameter.
- Explain the combinatorial calculation of $\mathcal{F}^{(s)}_{\mathbb{B}}$
  if $\mathbb{B}$ is the classifying space of a homotopy finite strict
  ω-groupoid (represented by a crossed complex of groupoids).
- Explain the construction of the extended version of $\mathcal{F}^{(s)}_{\mathbb{B}}$.

Homotopy finite spaces (or simplicial sets)

Definition (Homotopy finite space)

A space X (or simplicial set X) is called homotopy finite (HF) if:

- X has only a finite number of path components.
- Let $K \in \pi_0(X)$, the set of path components of X.
  There exists an $n \in \mathbb{N}$ such that:
  - $\pi_i(K)$ is trivial, if $i > n$.
  - $\pi_i(K)$ finite, if $i = 1, \ldots, n$.

So X has only a finite number of non-trivial homotopy groups, all of which are finite.

Definition (n-type)

Let $n \in \mathbb{N}$. A path-connected space X is called an n-type if:

1. X is homeomorphic to a CW-complex
   (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

An n-type is HF if all of its homotopy groups are finite.

Classifying spaces of groups, etc

Example

If G is a a finite group, then its classifying space B_G is path-connected and satisfies:
$\pi_1(B_G, *) \cong G$, and $\pi_i(B_G, *) = 0, \text{ if } i \geq 2.$
So B_G is a finite 1-type. So B_G is a HF space.

The homotopy category of 1-types, with maps $X \to Y$ continuous maps up to homotopy, is equivalent to the category of groups and group homomorphisms, considered up to conjugation.

More generally, if G is a finite groupoid, or finite 2-group, then B_G is homotopy finite
More examples later.

The homotopy content of a homotopy finite space

Definition (Homotopy content)

If X is HF, the *homotopy content* of X is:

$$\chi^{\pi}(X) = \sum_{K \in \pi_0(X)} \frac{|\pi_2(K)|\ |\pi_4(K)|\ |\pi_6(K)| \dots}{|\pi_1(K)|\ |\pi_3(K)|\ |\pi_5(K)| \dots} \in \mathbb{Q}$$

Example (Classifying spaces of finite groups)

If G is a finite group then $\chi^{\pi}(B_G) = 1/|G|$.

The homotopy content first appeared (I think) in:
Frank Quinn. Lectures on axiomatic topological quantum field theory. In Geometry and quantum field theory. (1995)

John C. Baez and James Dolan. From finite sets to Feynman diagrams. In Mathematics unlimited-2001 and beyond (2001)

Some properties of the homotopy content

- If X and Y are HF then so are $X \times Y$ and $X \sqcup Y$, and:

$$\chi^\pi(X \times Y) = \chi^\pi(X) \times \chi^\pi(Y)$$

$$\chi^\pi(X \sqcup Y) = \chi^\pi(X) + \chi^\pi(Y)$$

- If $p\colon E \to B$ is a fibration of HF spaces, and B path-connected, $b \in B$, $F_b = p^{-1}(b)$:

$$\chi^\pi(E) = \chi^\pi(B) \times \chi^\pi(F_b)$$

- If M is a compact CW-complex, $\mathbb{B}$ is HF space. Then the function space below is HF:

$$\mathrm{TOP}(M, \mathbb{B}) = \{f\colon M \to \mathbb{B} \mid f \text{ is continuous}\}.$$

  In particular if M is a compact smooth manifold.

Quinn's (finite total homotopy) TQFT

Let $\mathbb{B}$ be a HF-space. Let $s \in \mathbb{C}$.
We define a functor: $\mathcal{F}_{\mathbb{B}}^{(s)} \colon \boldsymbol{(n, n+1)}\textbf{-Cob} \to \textbf{Vect}$.

- If A is an n-manifold then:

$$\mathcal{F}_{\mathbb{B}}^{(s)}(A) = \mathbb{C}([A, \mathbb{B}]) = \mathbb{C}\big(\pi_0(\mathrm{TOP}(A, \mathbb{B})\big).$$

- Matrix elements assigned to cobordisms $A \xrightarrow{\;i\;} M \xleftarrow{\;j\;} B$

$$\Big\langle [f] \Big| \mathcal{F}_{\mathbb{B}}^{(s)}(M) \Big| [f'] \Big\rangle = \chi^{\pi} \Big\{ H \colon M \to \mathbb{B} : \quad \begin{array}{ccccc} A & \xrightarrow{\;i\;} & M & \xleftarrow{\;j\;} & B \\ & {\scriptstyle f} \searrow & \downarrow{\scriptstyle H} & \swarrow {\scriptstyle f'} & \\ & & \mathbb{B} & & \end{array} \Big\}$$

$$\times \big(\chi^{\pi}(\mathrm{PC}_f(\mathrm{TOP}(A, \mathbb{B})))\big)^{s} \; \big(\chi^{\pi}(\mathrm{PC}_{f'}(\mathrm{TOP}(B, \mathbb{B})))\big)^{1-s}$$

Discussion

Note: Quinn TQFT $\mathcal{F}_{\mathbb{B}}^{(s)}$ can be twisted by classes in $H^{n+1}(\mathbb{B}, U(1))$.

- Let G be a finite group. Let $\mathbb{B}$ be the classifying space of G. Then $\mathcal{F}_{\mathbb{B}}^{(s)}$ coincides with Dijkgraaf-Witten TQFT.

  Explicitly calculable. Related to topological gauge theory.
  Related to Kitaev quantum double model.

- Let $\mathcal{G}$ be a finite 2 group. Let $\mathbb{B}$ be the classifying space of $\mathcal{G}$. $\mathcal{F}_{\mathbb{B}}^{(s)}$ coincides with (twisted) Yetter TQFT (___ / Porter).

  Explicitly calculable. Related to topological higher gauge theory.
  Related to higher Kitaev models formulated with 2-groups.

- (Ellis Theorem) Any HF space is homotopic to a space of the form $|\overline{W}(G)|$, where G is a finite simplicial group.

  Let $\mathbb{B} = \overline{W}(G)$. *If* $M\colon A \to B$ *is a triangulated cobordism, we can compute* $\mathcal{F}_{\mathbb{B}}^{(s)}(M\colon A \to B)$ *using simplicial homotopy tools.*

  All Quinn's TQFTs $\mathcal{F}_{\mathbb{B}}^{(s)}$ therefore are 'combinatorial'.

Crossed modules (as models for homotopy 2-types)

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an <u>abelian group</u>.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.

- $\partial\colon A \to G$, map of <u>abelian groups</u>. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed complexes (of groupoids)

A crossed complex is given by a complex

$$\mathcal{C} := \ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} C_3 \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1$$

of groupoids, all with object set C_0. Such that:

- All groupoids for C_i, $i \geq 2$ are totally disconnected.
- All boundary maps are the identity over the object C_0.
- We have an action of C_1 over on all groupoids C_i, $i \geq 2$
- All boundary maps preserve the action.
- Peiffer 1: If $x \xrightarrow{g} y \in C_1$ and $K \in C_n(y, y)$ then (for $k \geq 2$):
$$\partial(g \triangleright K) = g\partial(K)g^{-1}$$
- Peiffer 2: If $K, L \in C_2(y, y)$ then $\partial(K) \triangleright L = KLK^{-1}$
- The action of $\partial(C_2)$ is trivial on all groupoids C_i for $i \geq 3$.
- C_i is abelian if $i \geq 3$.

Monoidal closed category Crs

The category Crs of crossed complexes is equivalent to the category of strict omega-groupoids (Brown–Higgins).

Also: Crs is a monoidal closed category (Brown–Higgins).

- Given $\mathcal{A}$ and $\mathcal{B}$ we can form tensor product $\mathcal{A} \otimes \mathcal{B}$.
- Given $\mathcal{A}$ and $\mathcal{B}$ we can form “function space” $\mathrm{CRS}(\mathcal{A}, \mathcal{B})$.
- Natural equivalence $\mathrm{Crs}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{Crs}(\mathcal{A}, \mathrm{CRS}(\mathcal{B}, \mathcal{C}))$.
- Given $\mathcal{A}$ and $\mathcal{B}$, $\pi_1(\mathrm{CRS}(\mathcal{A}, \mathcal{B}))$ is groupoid of maps $\mathcal{A} \to \mathcal{B}$ and (2-fold homotopy classes of) homotopies between them.

Example

Let G and H be finite groups, seen as a crossed complex

- $G \otimes H$ is the free product $G * H$.
- $\mathrm{CRS}(G, H)$ is the groupoid with
  - objects maps $f \colon G \to H$.
  - morphisms $f \xrightarrow{h} f'$ are elements of H conjugating f into f'.

Fundamental crossed complexes and nerves

Theorem (Brown-Higgins)

Let X be a CW-complex. Then the sequence

$$\Pi(X) := \ldots \xrightarrow{\partial} \pi_n(X^n, X^{n-1}, X^0) \xrightarrow{\partial} \pi_{n-1}(X^{n-1}, X^{n-2}, X^0) \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} \pi_2(X^2, X^1, X^0) \xrightarrow{\partial} \pi_1(X^1, X^0)$$

is a totally free crossed complex with object set X_0.

Definition (Nerve and classifying space of crossed complexes)

The nerve $\mathcal{NC}$ of the crossed complex

$$\mathcal{C} = \ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1$$

is the simplicial sets given by all maps $\Pi(\Delta(n)) \to \mathcal{C}$.
The classifying space of $\mathcal{C}$ is $\mathbb{B}_{\mathcal{C}} := |\mathcal{NC}|$.

$\mathcal{C}$-colourings ($\mathcal{C}$ a crossed complex)

Let M be a manifold with triangulation t.
Let M_t be corresponding CW-complex. Consider:

$$\mathcal{C} = \ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} C_2 \xrightarrow{\partial} C_1$$

Maps $f\colon \Pi(M_t) \to \mathcal{C}$ are in 1-to-1 correspondence with $\mathcal{C}$-colorings:

- a map $f_0\colon \text{Vertices}(M_t) \to C_0$
- a map $f_1\colon \text{edges}(M_t) \to C_1$, looking like:

$$f_0(v_0) \xrightarrow{f_1(\gamma)} f_0(v_1) \text{ at each edge } v_0 \xrightarrow{\gamma} v_1 \text{ of } M_t$$

- a map $f_2\colon \text{triangles}(M_t) \to C_2$, looking like:

$$\begin{array}{ccc} f_0(v_0) & \xrightarrow{f_1(\gamma_{01})} & f_0(v_1) \\ & f_2(\Delta_{012}) & \\ {\scriptstyle \partial(f_2(\Delta_{123}))f_1(\gamma_{02})} \searrow & & \swarrow {\scriptstyle f_1(\gamma_{12})} \\ & f_0(v_2) & \end{array} \quad \text{at each triangle} \quad \begin{array}{ccc} v_0 & \xrightarrow{\gamma_{01}} & v_1 \\ & \Delta_{012} & \\ {\scriptstyle \gamma_{02}} \searrow & & \swarrow {\scriptstyle \gamma_{12}} \\ & v_2 & \end{array}$$

Rule: boundary of element associated to a $n+1$-simplex is a twisted product of the elements associated to the simplexes in the boundary of the $n+1$-simplex.

Calculation of Quinn's $\mathcal{F}^{(s)}_{\mathbb{B}}$ for $\mathbb{B} = |\mathcal{N}(\mathcal{C})|$.

Let $\mathcal{C}$ be a pointed homotopically finite crossed complex
Hence $\mathbb{B} := |\mathcal{N}\mathcal{C}|$ is a homotopically finite space.

Theorem (___ /Porter (following Brown-Higgins))

Let A be a closed n-manifold with a triangulation t. Then

$$\mathcal{F}^{(s)}_{\mathbb{B}}(A) \cong \mathbb{C}(\pi_0(\mathrm{CRS}(\Pi(A_t), \mathcal{C})).$$

Here $\mathrm{CRS}(\_, \_)$ is internal-hom in the cat. of crossed complexes.

In particular a basis of $\mathcal{F}^{(s)}_{\mathbb{B}}(A)$ consists of homotopy classes of crossed complex maps $f : \Pi(A_t) \to \mathcal{C}$,
considered up to homotopy / pseudonatural equivalence.

Note $\mathrm{CRS}(\Pi(A_t), \mathcal{C})$ is the crossed complexes of all maps $f : \Pi(A_t) \to \mathcal{C}$ and their homotopies / natural transformations of all orders.

Calculation of Quinn $\mathcal{F}_{\mathbb{B}}^{(s)}$ for $\mathbb{B} = |\mathcal{N}(\mathcal{C})|$.

Consider a cobordism

$$A \xrightarrow{\ i\ } M \xleftarrow{\ j\ } B$$

Consider a triangulation t of the cobordism. (A triangulation M_t of M, inducing triangulations A_t and B_t, of A and B):

Theorem (___/Porter (following Brown-Higgins))

Given $f\colon \Pi(A_t) \to \mathcal{C}$ *and* $f'\colon \Pi(A_t) \to \mathcal{C}$

$$\Big\langle [f] \Big| \mathcal{F}_{\mathbb{B}}^{(s)}(M) \Big| [f'] \Big\rangle = \#\Big\{ H\colon \Pi(M_t) \to \mathcal{C} : \begin{array}{ccccc} \Pi(A_t) & \xrightarrow{i} & \Pi(M_t) & \xleftarrow{j} & \Pi(B_t) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle f'} & \\ & & \mathbb{C} & & \end{array} \Big\}$$

$\times$*factors depending only on number of simplices of* A_t, B_t, M_t; *and* $\mathcal{C}$.

Extended cobordisms

Let $(\boldsymbol{n}, \boldsymbol{n}+\mathbf{1}, \boldsymbol{n}+\mathbf{2})$-**Cob** be the bicategory with:

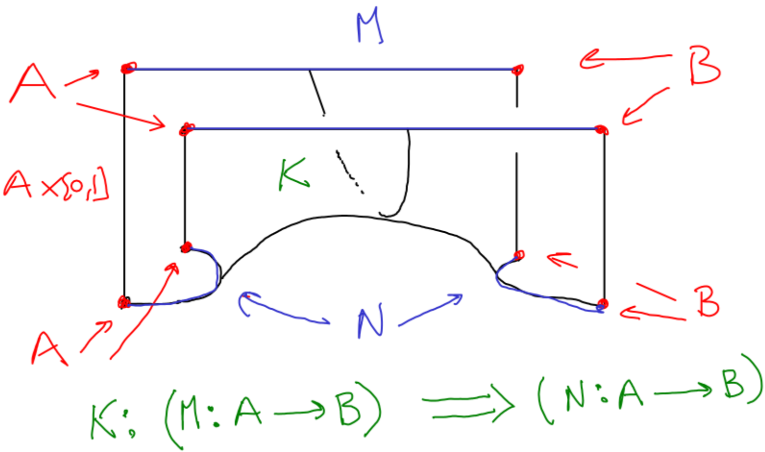
- Objects n-dimensional closed smooth manifolds $A, B, \ldots$
- 1-morphisms $M\colon A \to B$ are $(n, n+1)$-cobordisms (no equivalence relation is applied now):

$$A \xrightarrow{\;i\;} M \xleftarrow{\;j\;} B\,.$$

- 2-morphisms $K\colon M \implies N$ are $(n, n+1, n+2)$-extended-cobordisms (up to equivalence):

$$\begin{array}{ccccc}
A & \xrightarrow{\;i_1\;} & M & \xleftarrow{\;j_1\;} & B \\
{\scriptstyle \iota_0^A}\downarrow & & \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle \iota_0^B} \\
A \times [0,1] & \xrightarrow{\;i_E\;} & K & \xleftarrow{\;i_W\;} & B \times [0,1] \\
{\scriptstyle \iota_1^A}\uparrow & & \uparrow{\scriptstyle i_S} & & \uparrow{\scriptstyle \iota_1^B} \\
A & \xrightarrow[\;i_2\;]{} & N & \xleftarrow[\;j_2\;]{} & B
\end{array}$$

Extended cobordisms



Horizontal and vertical composition is performed via pushouts.
(As before, we need collars to construct smooth structures.)

(Once)-Entended TQFTs

A once-extended TQFT is a symmetric monoidal bifunctor:

$$\mathcal{F}\colon \boldsymbol{(n, n+1, n+2)}\text{-}\mathbf{Cob} \to \mathbf{Alg}.$$

Here $\mathbf{Alg}$ is some "algebraic" symmetric monoidal bicategory.

In this talk we will take:

- $\mathbf{Alg}$ to be the bicategory $\mathbf{Mor}$ with:
  - objects algebras $\mathcal{A}, \mathcal{B}$,...
  - with 1-morphisms $\mathcal{M}\colon \mathcal{A} \to \mathcal{B}$ being $(\mathcal{A}, \mathcal{B})$-bimodules $\mathcal{M}$.
  - Composition $\mathcal{A} \xrightarrow{\mathcal{M}} \mathcal{B} \xrightarrow{\mathcal{N}} \mathcal{C}$ is $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}$.
  - 2-morphisms $(\mathcal{A} \xrightarrow{\mathcal{M}} \mathcal{B}) \implies (\mathcal{A} \xrightarrow{\mathcal{M}'} \mathcal{B})$ are bimodule maps $\mathcal{M} \to \mathcal{M}'$.
- We will also consider $\mathbf{Alg}$ to be the bicategory $\mathbf{Prof}$ with
  - objects groupoids $\mathcal{G}$, $\mathcal{H}$,
  - 1-morphisms $\mathcal{G} \to \mathcal{H}$ are (enriched) profunctors, i.e. functors $\mathcal{G}^{\mathrm{op}} \times \mathcal{H} \to \mathbf{Vect}$. Composition is via coends.
  - 2-Morphisms are natural transformations of functors.

Extended version of Quinn's TQFT

Let $\mathbb{B}$ a HF space. There is an extended TQFT $\widehat{\mathcal{Q}}_{\mathbb{B}}$ such that:

- An n manifold A is sent to the fundamental groupoid of $\mathrm{TOP}(A, \mathcal{B})$, so $\widehat{\mathcal{Q}}_{\mathbb{B}}(A) = \pi_1(\mathrm{TOP}(A, \mathbb{B}), \mathrm{TOP}(A, \mathbb{B}))$.
- A cobordism:

  $$A \xrightarrow{i} M \xleftarrow{j} B\,.$$

  is sent to the profunctor:

  $$\widehat{\mathcal{Q}}_{\mathbb{B}}(M) \colon \widehat{\mathcal{Q}}_{\mathbb{B}}(A)^{\mathrm{op}} \times \widehat{\mathcal{Q}}_{\mathbb{B}}(B) \to \mathbf{Vect}$$

  obtained from the path-space fibration:

  $$\langle i^*, j^* \rangle \colon \mathrm{TOP}(M, \mathbb{B}) \to \mathrm{TOP}(A, \mathbb{B}) \times \mathrm{TOP}(B, \mathbb{B}).$$

  So given $f \colon A \to \mathbb{B}$ and $f' \colon A \to \mathbb{B}$
  $\widehat{\mathcal{Q}}_{\mathbb{B}}(M)(f, g) = \pi_0(\{H \colon M \to \mathbb{B} \mid H \circ i = f \text{ and } H \circ j = f'\})$.
- Natural transformations of profunctors $\widehat{\mathcal{Q}}_{\mathbb{B}}(M) \implies \widehat{\mathcal{Q}}_{\mathbb{B}}(N)$ can be associated to extended cobordism $K \colon M \implies N$.

Decorated manifolds

Let $\mathbb{B}$ be a homotopy finite space.

Definition (Decorated manifold)

A decorated manifold $\mathbf{A} = (A, \overline{x}_A)$ is a manifold A together with a finite subset $\overline{x}_A$ of $\mathrm{TOP}(A, \mathbb{B})$, containing at least one element for each path component of $\mathrm{TOP}(A, \mathbb{B})$.

There is a bicategory $(\boldsymbol{n}, \boldsymbol{n}+\mathbf{1}, \boldsymbol{n}+\mathbf{2})$-$\overline{\mathbf{Cob}}$ of decorated manifolds, (undecorated) cobordisms and (undecorated) extended cobordism.

Finitary extended version of Quinn's TQFT

Let $\mathbb{B}$ be a homotopy finite space.

Theorem (Finitary extended Quinn TQFT)

We have an extended TQFT:

$$\overline{\mathcal{Q}}_{\mathbb{B}} \colon \boldsymbol{(n, n+1, n+2)}\text{-}\overline{\mathbf{Cob}} \to \mathbf{Prof},$$

sending $\mathbf{A} = (A, \overline{x}_A)$ *to* $\pi_1(\mathrm{TOP}(A, \mathbb{B}), \overline{x}_A)$*.*

Note: $\overline{\mathcal{Q}}_{\mathbb{B}}(\mathbf{A})$ *is a finite groupoid since* $\overline{x}_A$ *is finite and* $\mathbb{B}$ *is homotopically finite. Previously the groupoid:*
$\overline{\mathcal{Q}}_{\mathbb{B}}(A) = \pi_1(\mathrm{TOP}(A, \mathbb{B}), \mathrm{TOP}(A, \mathbb{B}))$ *had infinite objects.*

The bifunctor $\overline{\mathcal{Q}}_{\mathbb{B}}$ *"linearises" to a bifunctor, also denoted*

$$\overline{\mathcal{Q}}_{\mathbb{B}} \colon \boldsymbol{(n, n+1, n+2)}\text{-}\overline{\mathbf{Cob}} \to \mathbf{Mor},$$

sending $\mathbf{A} = (A, \overline{x}_A)$ *to groupoid algebra* $\mathbb{C}(\pi_1(\mathrm{TOP}(A, \mathbb{B}), \overline{x}_A))$*.*

The case of crossed complexes / strict omega-groupoids

Suppose that $\mathbb{B} = \mathbb{B}_{\mathcal{C}}$, where $\mathcal{C}$ is a crossed complex.

Theorem

If A has a triangulation A_t then A is naturally decorated. Moreover:

$$\overline{\mathcal{Q}}_{\mathbb{B}}(A) = \mathbb{C}(\pi_1(\mathrm{CRS}(\Pi(A_t), \mathcal{C})))$$

Note: $\pi_1(\mathrm{CRS}(\Pi(A_t), \mathcal{C}))$ *is the groupoid of crossed complex maps $\Pi_1(A_t) \to \mathcal{C}$, and (2-fold homotopy classes of) homotopies between them.*

This permits the computation of the extended Quinn TQFT $\overline{\mathcal{Q}}_{\mathbb{B}}(A)$ if $\mathbb{B}$ is the classifying space of a finite crossed complex.

Example

Let G be a finite group. $\mathbb{B} = B_G$.
Some examples of $\overline{\mathcal{Q}}_{\mathbb{B}} \colon \boldsymbol{(n, n+1, n+2)}\text{-}\overline{\mathbf{Cob}} \to \mathbf{Mor}$:

- $n = 0$, then $\overline{\mathcal{Q}}_{\mathbb{B}}(.) = \mathbb{C}(G)$
- $n = 1$, then $\overline{\mathcal{Q}}_{\mathbb{B}}(S^1)$ is the quantum double of $\mathbb{C}(G)$.