

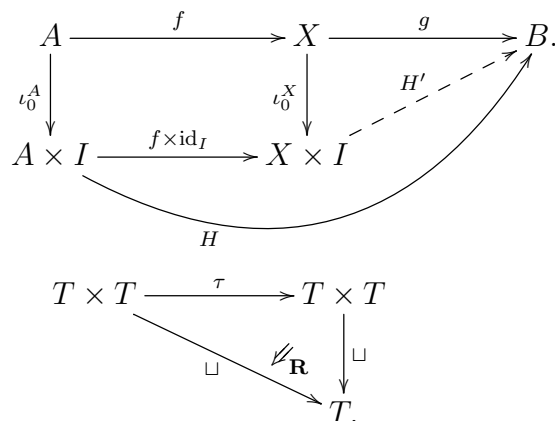
Sample HTML file: produced with `lwarp`, with mathematical formulae displayed with `MathJax`, and `xymatrix` commutative diagrams displayed with `XyJax-v3`

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The responsibility for this HTML file is João Faria Martins' only. Corrections, suggestions, etc. are very welcome, and can be sent to: [j.fariamartins@leeds.ac.uk](mailto:j.fariamartins@leeds.ac.uk).

**This is strictly a non-specialist experimental file.**

See the links below for:

- “stand-alone” version of this webpage,
- “split” version of this webpage (with sans serif fonts),
- pdf version of this page,
- Latex source for the stand alone version of this page `Sample_Lwarp_Xy-jax.tex`,
- Latex source for the split version of this page, with sans serif fonts: `Sample_Lwarp_Xy-jax-split.tex`,
- CSS files for the stand-alone version of this page: `lwarp.css` (automatically provided by *lwarpmk*) and `my_file_stand-alone.css` (minor tweak to adjust margins),
- CSS files for the split version of this page: `lwarp_formal.css` (automatically provided by *lwarpmk*) and `my_file_split.css` (minor tweak enforcing sans serif fonts).
- Configuration file required for browsers to find Xy-Jaxv6, `lwarp-with-Xy-jax_v6.txt`. This script is essentially the original Lwarp MathJax emulation code, `lwarp_mathjax.txt` (automatically created by `lwarp v0.902`), with two minimal additions, as shown. File tweaked 25May2026. Requires `xypic.js` in directory “js”.

## 1 Context and instructions: using Lwarp to compile latex to HTML, so that *xymatrix* diagrams are displayed with Xy-Jax

This HTML file (that you are reading) was produced with the `lwarp.sty` package, available with **TeXLive**.<sup>1</sup> In particular the preamble contains:

```
\documentclass[a4paper,12pt]{article}
\usepackage[mathjax]{lwarp}
```

The version of the software was:

```
pdfTeX, Version 3.141592653-2.6-1.40.24 (TeX Live 2022) (preloaded format=pdflatex)
lwarpmk: lwarpmk: v0.910 Automated make for the LaTeX Lwarp package.
```

The source latex file can be found here: `Sample_Lwarp_Xy-jax.tex` or here `Sample_Lwarp_Xy-jax-split.tex`, depending whether you are seeing the “stand-alone” version of this webpage, or the “split” version of this webpage.

This HTML file also makes use of the following configuration script, so that **Xy-pic** diagrams, particularly *xymatrix* commutative diagrams, can be displayed by using **Xy-Jax-v3**. (Or at least a good sample of them: there seem to be some compatibility issues.)

---

<sup>1</sup>See also <https://people.bath.ac.uk/feb/lwarp/lwarp-intro.html>, for a quick introduction on how to use Lwarp, and without which this explanation that you are reading would not have been written.

lwarp-with-Xy-jax\_v6.txt.

This script file has minimal additions (two in total) in the original Lwarp MathJax emulation code, that is automatically created by lwarp v0.902), namely:

lwarp\_mathjax.txt,

This modification was produced under items 5 and 6 of the licence <https://www.ctan.org/license/lppl1.3> (The LATEX Project Public License 1.3).

Credit is due to instructions in: <https://github.com/sonoisa/XyJax-v3> (the official Xy-Jaxv3 documentation) and in "40.11 lwarp\_mathjax.txt" of the official Lwarp documentation.

If using Xy-pic diagrams displayed with **XyJax-v3**, insert the following in the preamble of your latex file:

```
\usepackage[all]{xy}
\MathJaxFilename{lwarp-with-Xy-jax_v6.txt}
```

Also create a directory "js", and copy the file xypic.js to it.

**Compile .tex to .html using:**

1. *pdflatex Sample\_Lwarp\_Xy-jax.tex* (possibly twice) : creates *Sample\_Lwarp\_Xy-jax.pdf* (usual pdf file, that can also be shared) and *Sample\_Lwarp\_Xy-jax\_html.tex*. (And some other aux files.)
2. *lwarpmk html* : converts *Sample\_Lwarp\_Xy-jax\_html.tex* to *Sample\_Lwarp\_Xy-jax\_html.html*. And then creates *Sample\_Lwarp\_Xy-jax.html* (possibly split into different HTML files if such option is chosen).

**Note:** Use *lwarpmk html1* to force a recompile.

**Note:** In case of error, with *lwarpmk html*, use *lwarpmk pdftohtml* to create HTML file.

**Note:** When using *bibtex* files, then the following is required:

- *bibtex* *Sample\_Lwarp\_Xy-jax\_html.aux*
- *pdflatex* *Sample\_Lwarp\_Xy-jax\_html.tex*

The following steps are only needed once (any time tikz figures change). Only needed with tikz.

3. *lwarpmk limages* : If there are images done with tikz, tikz-cd, etc.
4. Repeat 2 (now using *lwarpmk html1*) and 3 (to finish creating the images).

**Note.** *It is advisable to use a recent version of **TeXLive**.*

**Warnings:**

- The usage of **XyJax-v3** software with lwarp is experimental.
- Not all accessibility features of commutative diagrams displayed with XyJax-v3 work. Among the accessibility features that seem to work well are:
  1. View original latex source.
  2. Zoom maths expression.

## 2 Examples: xymatrix diagrams displayed with Xy-Jaxv3 using Lwarp

This is an example of a HTML file with:

- Mathematical formulae displayed with **MathJax**:

$$x^2 + y^2 = 1, \tag{1}$$

$$(x \xrightarrow{\gamma} y) \in \pi_1(X, X_0),$$

$$\cdots \rightarrow \pi_i(E_b, x) \xrightarrow{\iota} \pi_i(E, x) \xrightarrow{\partial} \pi_i(B, b) \xrightarrow{\delta} \pi_{i-1}(E_b, x) \rightarrow \cdots \xrightarrow{\iota} \pi_1(E, x) \xrightarrow{\partial} \pi_1(B, b) \xrightarrow{\delta_x} \pi_0(E_b) \xrightarrow{\iota} \pi_0(\dots)$$

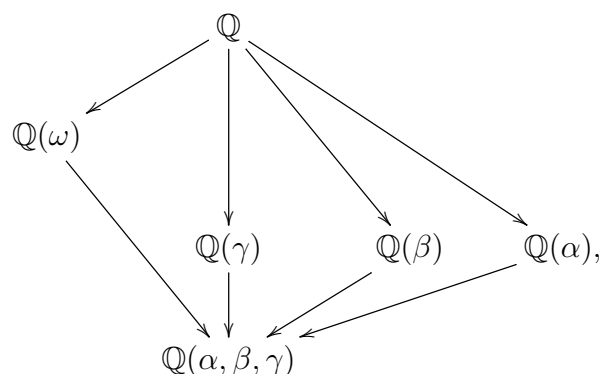
- To see how lecture notes look like with Lwarp, see subsections 6.1 and 6.2.
- This configuration is compatible with AMS-CD commutative diagrams, e.g. this diagram from the amscd package manual:


$$\begin{array}{ccc} A & \xrightarrow{a} & B \\ & \underset{b}{\downarrow} & \\ l \downarrow r & & l \uparrow r \\ C & \xleftarrow{a} & D \\ & \underset{b}{\downarrow} & \end{array}$$

- Commutative diagrams drawn with **Xy-pic** displayed by using **XyJax-v3**.  
For a good introduction to typesetting diagrams with Xy-pic see e.g.
  - [https://www.math.purdue.edu/~adebray/lecture\\_notes/using\\_xy.pdf](https://www.math.purdue.edu/~adebray/lecture_notes/using_xy.pdf) (Arun Debray Using Xy to typeset automata. July 5, 2016.)
  - <https://web.math.ku.dk/~holm/download/xyrefer.pdf> (Kristoffer H. Rose & Ross Moore: Xy-pic Reference Manual, Version 3.7 1999/02/16.)
  - See also the more recent user guide (Kristoffer H. Rose, Version 3.8.9, October 6, 2013) in the official page of xymatrix – Commutative diagrams using XY-pic

E.g.

$$\begin{array}{ccc} S_3 & \xrightarrow{f} & S_4 \\ & \searrow \cong & \downarrow \pi \\ & & S_4/V, \end{array}$$



(123). 

$$\begin{array}{ccccc}
 A & \xrightarrow{f} & X & \xrightarrow{g} & B, \\
 \downarrow \iota_0^A & & \downarrow \iota_0^X & \nearrow H' & \\
 A \times I & \xrightarrow{f \times \text{id}_I} & X \times I & & \\
 & \searrow H & & & 
 \end{array}$$

$$\begin{array}{ccc}
 \bigoplus_{y \in H_0} \mathbf{H}_1(x, y) \otimes \mathbf{H}'_1(y, z) & \xrightarrow{\text{proj}} & \int^{y \in H_0} \mathbf{H}_1(x, y) \otimes \mathbf{H}'_1(y, z) \\
 \downarrow \bigoplus_{y \in H_0} \eta(x, y) \otimes \eta'(y, z) & & \downarrow \int^{y \in Y} \eta(x, y) \otimes \eta'(y, z) \\
 \bigoplus_{y \in H_0} \mathbf{H}_2(x, y) \otimes \mathbf{H}'_2(y, z) & \xrightarrow{\text{proj}} & \int^{y \in H_0} \mathbf{H}_2(x, y) \otimes \mathbf{H}'_2(y, z),
 \end{array}$$

$$\begin{array}{ccc}
 \int^{y \in Y} \overline{\mathbf{H}}_{(X, Y)}^{M_1}(-, y) \otimes \overline{\mathbf{H}}_{(Y, Z)}^{M_2}(y, -) & \xrightarrow{\eta_{(X, Y, Z)}^{M_1, M_2}} & \overline{\mathbf{H}}_{(X, Z)}^{M_1 \times_Y M_2} \\
 \parallel \mathbf{2H}_{(X, Y)}^{W_1} \bullet \mathbf{2H}_{(Y, Z)}^{W_2} & & \parallel \mathbf{2H}_{(X, Z)}^{W_1 \#_0 W_2} \\
 \int^{y \in Y} \overline{\mathbf{H}}_{(X, Y)}^{N_1}(-, y) \otimes \overline{\mathbf{H}}_{(Y, Z)}^{N_2}(y, -) & \xrightarrow{\eta_{(X, Y, Z)}^{N_1, N_2}} & \overline{\mathbf{H}}_{(X, Z)}^{N_1 \times_Y N_2}.
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
 & \Downarrow \eta & \\
 \mathcal{C} & \xrightarrow{F'} & \mathcal{D}
 \end{array}$$

$$\begin{array}{ccc}
 T \times T & \xrightarrow{\tau} & T \times T \\
 & \searrow \sqcup & \downarrow \sqcup \\
 & & T.
 \end{array}$$

$$\begin{array}{ccc}
 \mathcal{C} \sqcup \mathcal{C}' & \xrightarrow{T_b \circ \langle f, f' \rangle} & \text{Grp.} \\
 & \Downarrow \langle \eta, \eta' \rangle & \\
 \mathcal{C} \sqcup \mathcal{C}' & \xrightarrow{\langle T, T' \rangle} & \text{Grp.}
 \end{array}$$

$$\begin{array}{ccc}
 \begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
 & \Downarrow \eta & \\
 \mathcal{C} & \xrightarrow{F''} & \mathcal{D} \\
 & \Downarrow \eta' & \\
 \mathcal{C} & \xrightarrow{F''} & \mathcal{D}
 \end{array} & = & \begin{array}{ccc}
 \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
 & \Downarrow \eta \#_1 \eta' & \\
 \mathcal{C} & \xrightarrow{F''} & \mathcal{D}
 \end{array}
 \end{array}$$

$$\begin{array}{ccccc}
(\mathbf{Cob}^{(n,n+1,n+2)})^3 & \xrightarrow{\mathcal{F}_B \times \mathcal{F}_B \times \mathcal{F}_B} & & & (\mathbf{Prof})^3 \\
\downarrow \text{id} \times \sqcup & \searrow \sqcup \times \text{id} & \Downarrow \chi' \times \mathcal{F}_B & & \downarrow \otimes \times \text{id} \\
(\mathbf{Cob}^{(n,n+1,n+2)})^2 & \xleftarrow{\alpha} & (\mathbf{Cob}^{(n,n+1,n+2)})^2 & \xrightarrow{\mathcal{F}_B \times \mathcal{F}_B} & (\mathbf{Prof})^2 \\
\downarrow \sqcup & \swarrow \sqcup & \swarrow \chi' & \searrow \otimes & \downarrow \otimes \times \text{id} \\
(\mathbf{Cob}^{(n,n+1,n+2)}) & \xrightarrow{\mathcal{F}_B} & \mathbf{Prof} & & (\mathbf{Prof})^2 \\
& & & & \swarrow \otimes \\
& & & & \mathbf{Prof}
\end{array}$$

$\xRightarrow{\omega'}$

$$\begin{array}{ccccc}
(\mathbf{Cob}^{(n,n+1,n+2)})^3 & \xrightarrow{\mathcal{F}_B \times \mathcal{F}_B \times \mathcal{F}_B} & & & (\mathbf{Prof})^3 \\
\downarrow \text{id} \times \sqcup & & \Downarrow \mathcal{F}_B \times \chi' & \swarrow \text{id} \times \otimes & \downarrow \otimes \times \text{id} \\
(\mathbf{Cob}^{(n,n+1,n+2)})^2 & \xrightarrow{\mathcal{F}_B \times \mathcal{F}_B} & (\mathbf{Prof})^2 & \swarrow \alpha & (\mathbf{Prof})^2 \\
\downarrow \sqcup & \Downarrow \chi' & \downarrow \otimes & \swarrow \otimes & \downarrow \otimes \times \text{id} \\
(\mathbf{Cob}^{(n,n+1,n+2)}) & \xrightarrow{\mathcal{F}_B} & \mathbf{Prof} & & (\mathbf{Prof})^2 \\
& & & & \swarrow \otimes \\
& & & & \mathbf{Prof}
\end{array}$$

### 3 Issues when using XyJax-v3 and Lwarp for typesetting xymatrix diagrams

Please do contact me if you know the solution to any of these issues. Remember: this is strictly a non-specialist file!

- The commands of the form:

`\stackrel{(*)}{\implies}`

or

`\left ( x \xrightarrow{f} y \right)`

do not seem to work well inside

`\begin{equation} \end{equation}.`

So use instead, for example,

`$$ \stackrel{(*)}{\implies} $$`

`$$ \left ( x \xrightarrow{f} y \right) $$`

. Here is how it looks:

$$\begin{array}{c}
(*) \\
\implies \\
\left( x \xrightarrow{f} y \right)
\end{array}$$

- Sometimes there are errors if there are mathematical commands in the title of sections, subsections, etc.
- Lwarpmk gives error messages if it finds *xymatrix* inside:

```
\begin{equation} \end{equation}.
```

(However a html file can still be produced with *lwarpmk pdftohtml*.)

So avoid:

```
\begin{equation} \xymatrix{ } \end{equation},
```

and use instead:

```
\[ \xymatrix{ } \],
```

or

```
$$ \xymatrix{ } $$.
```

- **Not all accessibility functions of MathJax work with Xy-Jaxv3, only some can be used, e.g. zoom and source latex code.**  
Accessibility features of Xy-jax are not officially supported.
- References to section, chapters etc, have glitches **if inside mathematics environments**.  
For instance, the links provided do not appear to work well. E.g.

$$\exp(x + yi) = e^x (\cos(y) + i \sin(y)) \quad (2)$$

$e^{i\pi} + 1 = (\cos(\pi) + i \sin(\pi)) + 1$ , using Equation (2). (Just for testing) equations (1), (3)  
= 0. A reference to second section 2

Outside maths environments all seems to work fine:

Using Equation (2). (Just for testing) equations (1), (3)  
A reference to second section 2

- When compiling xymatrix, or tikz, diagrams as sgv figures, with Alt-Text (which sometimes is necessary) it sometimes happens that the figure is not cropped correctly. This seems to depend strongly on the operating system being used.

## 4 More general notes

- **pdf** and **HTML** automatically coexist: here is the pdf version of this HTML file `Sample_Lwarp_Xy-jax.pdf`
- It is possible to split an HTML page into sub-pages: credit <https://people.bath.ac.uk/feb/lwarp/lwarp-intro.html>. Instructions can be seen in the source latex file of this HTML file `Sample_Lwarp_Xy-jax.tex`.

- Additions are required in the .tex file in order that **MathJax** displays commands, macros and definitions correctly. E.g. write:

```
\DeclareMathOperator{\Sym}{Sym}
\def \Mon {{\mathbf{Mon}}}
\newcommand{\Q}{\mathbb{Q}}
```

and then:

```
\begin{warpHTML}
  \CustomizeMathJax{\DeclareMathOperator{\Sym}{Sym}}
  \CustomizeMathJax{\def \Mon {{\mathbf{Mon}}}}
  \CustomizeMathJax{\newcommand{\Q}{\mathbb{Q}}}
\end{warpHTML}
```

This means that you can use slightly different versions of commands for pdf and for html.

- If using Xy-pic diagrams (if displayed as Xy-jax) put in the preamble of the tex file:

```
\MathJaxFilename{lwrap-with-Xy-jax_v6.txt}
```

You will also need this configuration file: lwrap-with-Xy-jax\_v6.txt. (Explanation is in Section 1. Also requires xypic.js in directory “js”)

- Sample use of Xy-pic compatible with lwrap (so that the resulting xymatrix code is readable by Xy-Jax): E.g.:

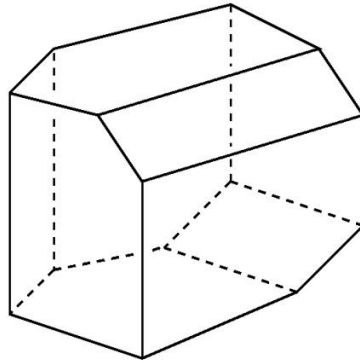
```
\[
\begin{matrix}
& \mathbb{Q} & & \mathbb{Q}(\omega) & & \\
& \mathbb{Q}(\omega) & \mathbb{Q} & \mathbb{Q}(\omega) & \mathbb{Q}(\omega) & \mathbb{Q}(\omega) \\
& \mathbb{Q}(\omega, \omega^2) & \mathbb{Q}(\omega) & \mathbb{Q}(\omega) & \mathbb{Q}(\omega) & \mathbb{Q}(\omega)
\end{matrix}
\]
```

or

```
\displaymathnormal{
\begin{matrix}
& S_3 & & S_4 & & \\
& S_3 & S_4 & S_4 & S_4 & S_4 \\
& S_4/V & & & & 
\end{matrix}
}
```

- Figures, with alternative text, can be included like this:

```
\includegraphics[width = 0.4\textwidth,
alt={Type your alternative text here}]{Stasheff.png}
```



- Figures, including **tikz** and **tikz-cd** figures, can be compiled as figures with **Alt Text**. (The same option is also available for xymatrix diagrams, and may be preferable to Xy-Jaxv3, in the cases when it is sensible to provide a comprehensive alternative text to the figure/diagram, instead of relying mainly on the availability of the latex code, for accessibility.)

**Warning:** In some operating systems, issues seem to exist with the conversion of tikz figures to svg: e.g. figures may be incorrectly cropped for the web-page. Size of xymatrix figures and tikz-cd figures seemingly then must be adjusted manually.

- When using **tikz** pictures use (note the Alt Text option):

```
\begin{figure}\ThisAltText{Alt text to your diagram}
\begin{tikzpicture}

\end{tikzpicture}
\end{figure}
```

or

```
\begin{center}\ThisAltText{Alt text to your diagram}
\begin{tikzpicture}

\end{tikzpicture}
\end{center}
```

Examples can be found towards the end of this file.

- When using **tikz-cd** use:

```
{\displaymathother
\ThisAltText{ Alt text to your commutative diagram }
\[
\begin{tikzcd} F(A) \arrow{r}{F(f)} & F(B) \\
G(A) \arrow{r}[swap]{G(f)} & \arrow{u}{\eta_A} & G(B) \\
& \arrow{u}[right]{\eta_B} & \\
\end{tikzcd}
\]
```

- It is possible to compile an **Xy-pic** diagram as a figure with alt text. Use:

```
{
\displaymathother
\ThisAltText{Write some alt text here.}
$$ \xymatrix{ } $$
}
```

Examples can be found towards the end of this file.

- **To select your own .CSS files use:**

```
\CSSFilename{your_file.css}
```

- Avoid good old Tex commands like

```
{\bf }, {\it }.
```

It seems that lwrap does not deal with them properly. Instead use:

```
\textbf{ }, \textit{ }.
```

(In fact better to avoid *italics* altogether for accessibility reasons: use **bold**.)

- Lwrap gives errors messages when compiling Xy-pic diagrams inside

```
\begin{equation} \end{equation}
```

Use instead:

```
[\xymatrix{} \]
or
\[\begin{xy} \xymatrix{} \end{xy}\]
```

Or use *lwrapmk pdftohtml* if there are compilation errors.

- Figures created with tikz and xymatrix frequently have issues: e.g. they may be too small (so size needs to be adjusted), or incorrectly cropped. Seems to depend on operating system.

## 5 Some examples of tikz and xymatrix diagrams compiled as svg images, with alternative text

As written in Section 2, it is possible to display xymatrix diagrams by using XyJax-v3. In some cases, it may however be preferable to compile xymatrix diagrams as figures with **Alt Text**. (E.g. it may be sensible to instead provide a comprehensive alternative text to the figure/diagram, instead of relying mainly on the availability of the latex code, for accessibility.)

This option (svg image with alternative text) is also available for tikz and tikz-cd figures. Examples are below.

### 5.1 An xymatrix diagram compiled as an image with alt text

An *xymatrix* diagram compiled into a picture with alt text. Can utilize equation numbers, with no errors. The size of the svg image must be adjusted. Sometimes the image is not correctly cropped: depends on operating system.

$$\begin{array}{ccc} T \times T & \xrightarrow{\tau} & T \times T \\ & \searrow & \downarrow \sqcup \\ & & T \end{array} \quad (3)$$

### 5.2 A tikz image compiled as an image with alt text:

Size can be adjusted. Sometimes (depending on operating system, it seems) the image may be incorrectly cropped. See figure 1.

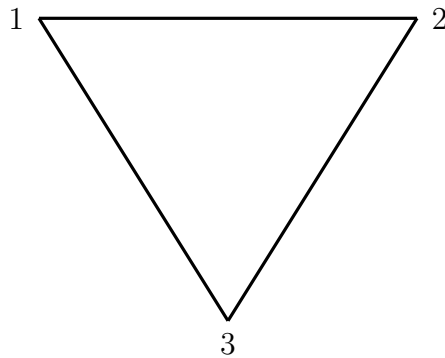
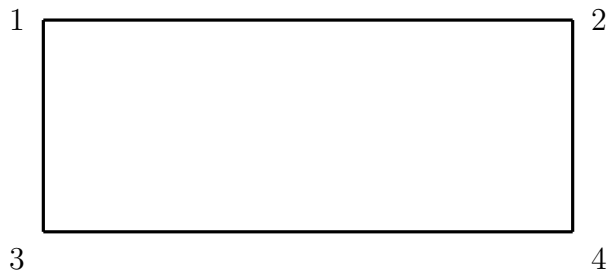


Figure 1: An equilateral triangle with its vertices labeled 1, 2 and 3

### 5.3 A tikz image compiled to an image with alt text, outside a float environment



## 5.4 A tikz image compiled to an image with alt text, outside a float environment, and with a caption

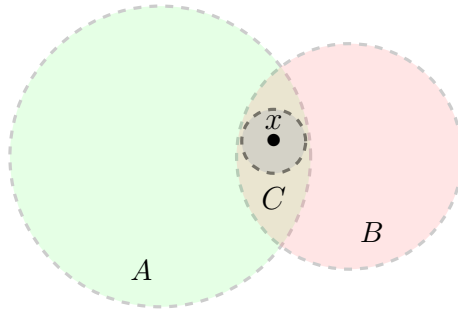


Figure 2: Given any  $A, B \in \mathcal{F}$ , if  $x \in A \cap B$  there exists  $C \in \mathcal{F}$  such that  $x \in C \subseteq A \cap B$ .

## 5.5 A tikz-cd diagram compiled into a figure with alt text:

Issues with size of diagrams created with tikz-cd. Sometimes (depending on operating system) the image may be incorrectly cropped.

The following diagram commutes.

$$\begin{array}{ccc}
 F(A) & \xrightarrow{F(f)} & F(B) \\
 \eta_A \uparrow & & \uparrow \eta_B \\
 G(A) & \xrightarrow{G(f)} & G(B)
 \end{array}$$

# 6 Sample mathematics

## 6.1 Adjoint functors and coadjoint functors via universal arrows

Let  $\mathcal{C}$  and  $\mathcal{D}$  be categories. Let  $G: \mathcal{D} \rightarrow \mathcal{C}$  be a functor.

**Definition 1.** Let  $A$  be an object of  $\mathcal{C}$ . A **universal arrow** from  $A$  to  $G: \mathcal{D} \rightarrow \mathcal{C}$  is a pair:

$$\left( F_A, A \xrightarrow{\eta_A} G(F_A) \right),$$

where  $F_A$  is an object in  $\mathcal{D}$  and  $\eta_A: A \rightarrow G(F_A)$  is an arrow in  $\mathcal{C}$ , such that the following universal property is satisfied:

For any object of  $B$  of  $\mathcal{D}$  and any arrow  $f: A \rightarrow G(B)$ , in  $\mathcal{C}$ , there exists a **unique** arrow  $\hat{f}: F_A \rightarrow B$ , in  $\mathcal{D}$ , that makes the following diagram, in  $\mathcal{C}$ , commute:

$$\begin{array}{ccc}
 A & \xrightarrow{\eta_A} & G(F_A) \\
 & \searrow f & \downarrow G(\hat{f}) \\
 & & G(B)
 \end{array}$$

**Exercise 2.** In the conditions of the previous definition, prove that if  $(F_A, A \xrightarrow{\eta_A} G(F_A))$  is a universal arrow from  $A$  to  $G$ , then we have a bijection:

$$\phi_{A,B}: \text{hom}_{\mathcal{C}}(A, G(B)) \longrightarrow \text{hom}_{\mathcal{D}}(F_A, B),$$

such that

$$(f: A \rightarrow G(B)) \xrightarrow{\phi_{A,B}} (\hat{f}: F_A \rightarrow B).$$

Moreover, prove that the bijection  $\phi_{A,B}$  is natural in  $B$ . This means that given any arrow  $g: B \rightarrow C$ , in  $\mathcal{D}$ , the following diagram (in the category of sets) commutes:

$$\begin{array}{ccc} \text{hom}_{\mathcal{C}}(A, G(B)) & \xrightarrow{\phi_{A,B}} & \text{hom}_{\mathcal{D}}(F_A, B) \\ m \mapsto G(g) \circ m \downarrow & & \downarrow n \mapsto g \circ n \\ \text{hom}_{\mathcal{C}}(A, G(C)) & \xrightarrow{\phi_{A,C}} & \text{hom}_{\mathcal{D}}(F_A, C) \end{array}$$

## 6.2 The Galois correspondence for $f(t) = t^3 - 2$ , over the rational field

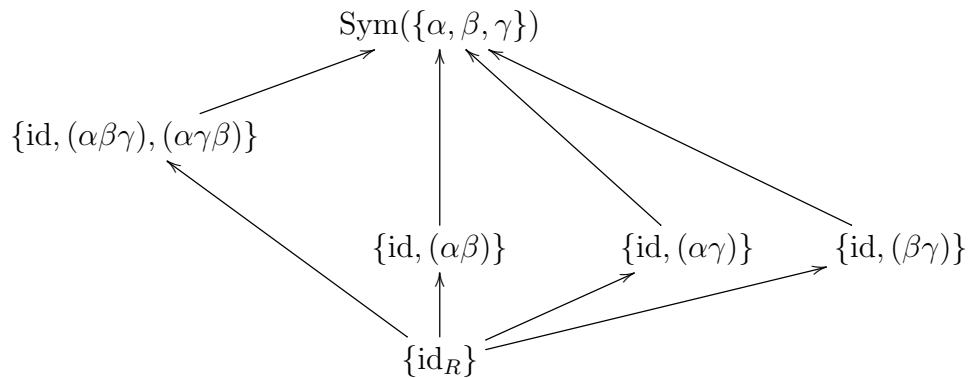
Let  $f(t) = t^3 - 2 \in \mathbb{Q}[t]$ . Let  $\omega = e^{\frac{2\pi i}{3}}$ . Let  $\alpha = \sqrt[3]{2}$ ,  $\beta = \alpha\omega$  and  $\gamma = \alpha\omega^2$ . Hence the set of roots of  $f$  is  $\{\alpha, \beta, \gamma\}$ . The splitting field of  $f$ , over  $\mathbb{Q}$  is:

$$\mathbb{Q}(\alpha, \beta, \gamma) = \mathbb{Q}(\alpha, \omega).$$

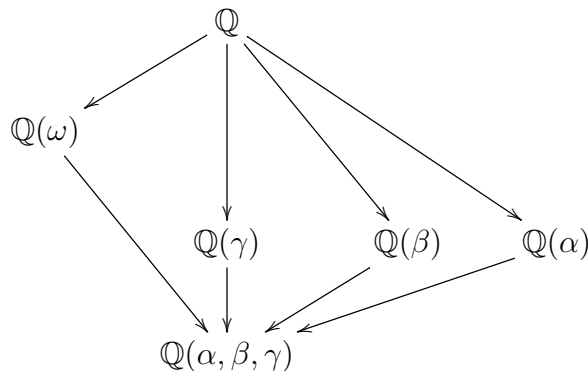
The monomorphism of groups,

$$\begin{aligned} \theta: \Gamma(f, \mathbb{Q}) &\longrightarrow \text{Sym}(\{\alpha, \beta, \gamma\}) \\ \tau &\longmapsto \left( \begin{array}{c} \{\alpha, \beta, \gamma\} \longrightarrow \{\alpha, \beta, \gamma\} \\ a \longmapsto \tau(a) \end{array} \right), \end{aligned}$$

is, in this case, an isomorphism. The diagram of subgroups of  $\Gamma(f, \mathbb{Q}) \cong \text{Sym}(\{\alpha, \beta, \gamma\})$  is below (note that inclusions go in the direction of arrows):



The corresponding diagram of intermediate fields  $\mathbb{Q} \subseteq L \subseteq \mathbb{Q}(\alpha, \beta, \gamma)$  is:



Also note that  $\{\text{id}_R\}$ ,  $\text{Sym}(R)$  and  $\{\text{id}, (\alpha\beta\gamma), (\alpha\gamma\beta)\} \cong A_3$  are the only normal subgroups of  $\text{Sym}(R)$ . So, respectively,

$$\mathbb{Q}(\alpha, \beta, \gamma) : \mathbb{Q}, \quad \mathbb{Q} : \mathbb{Q} \quad \text{and} \quad \mathbb{Q}(\omega) : \mathbb{Q} \quad (4)$$

are the only normal extensions of  $\mathbb{Q}$  contained in  $\mathbb{Q}(\alpha, \beta, \gamma)$ . Note that the fact that the extensions in (4) are normal also follows from the fact that they are the splitting fields of  $p(t) = t^3 - 2$ ,  $q(t) = t$  and  $r(t) = t^2 + t + 1$  (which has  $\omega$  and  $\omega^2$  as roots), over  $\mathbb{Q}$ .

**Remark 3.** Also note that we have a series of subfields of  $\mathbb{C}$  (we use  $\leq$  to denote subfield):

$$\mathbb{Q} \leq \mathbb{Q}(\omega) \leq \mathbb{Q}(\alpha, \beta, \gamma) = \mathbb{Q}(\alpha, \omega).$$

Note that:

- $\mathbb{Q}(\omega) : \mathbb{Q}$  is a normal extension (since it is the splitting field of  $t^2 + t + 1$ , over  $\mathbb{Q}$ ).
- $\mathbb{Q}(\omega, \alpha) : \mathbb{Q}(\omega)$  is also a normal extension. This is because  $\mathbb{Q}(\omega, \alpha)$  is the splitting field of  $t^3 - 2$  over  $\mathbb{Q}(\omega)$ .

And then it follows that:

- $\Gamma(\mathbb{Q}(\omega, \alpha), \mathbb{Q}(\omega))$  is normal in  $\Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q})$ ,
- we have a series of subgroups of  $\Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q})$ :

$$\{e\} = \Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q}(\omega, \alpha)) \trianglelefteq \Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q}(\omega)) \trianglelefteq \Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q}),$$

- The quotient groups can be explicitly determined:

$$- \Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q}) / \Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q}(\omega)) \cong \Gamma(\mathbb{Q}(\omega) : \mathbb{Q}) \cong \mathbb{Z}_2.$$

Where the last equation follows since  $\mathbb{Q}(\omega) : \mathbb{Q}$  is a normal extension of degree 2, since it is the splitting field of the irreducible polynomial  $t^2 + t + 1$ , over  $\mathbb{Q}$ . This  $\mathbb{Z}_2$  is an abelian group.

$$- \Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q}(\omega)) / \Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q}(\omega, \alpha)) \cong \Gamma(\mathbb{Q}(\omega, \alpha) : \mathbb{Q}(\omega)) \cong \mathbb{Z}_3.$$

Where the last equation follows since  $\mathbb{Q}(\omega, \alpha) : \mathbb{Q}(\omega)$  is a normal extension of degree 3. This is because it is the splitting field of the polynomial  $t^3 - 2$ , which is irreducible over  $\mathbb{Q}(\omega)$ . This is an abelian group.

What was just shown is a general pattern that exists any time we compute the splitting field of a polynomial that is soluble by radicals.