Quantum Topology and The Lorentz Group

by João Nuno Gonçalves Faria Martins

Thesis submitted to the University of Nottingham for the degree of Doctor of Philosophy, July 2004

Contents

Abstract

We analyse the perturbative expansion of knot invariants related with infinite dimensional representations of $\mathfrak{sl}(2,\mathbb{R})$ and the Lorentz group taking as a starting point the Kontsevich Integral and the notion of central characters of infinite dimensional unitary representations of Lie Groups. The prime aim is to define $\mathbb{C}$-valued knot invariants. This yields a family of $\mathbb{C}[[h]]$-valued knot invariants contained in the Melvin-Morton expansion of the Coloured Jones Polynomial. It is verified that for some knots, namely torus knots, the power series obtained have a zero radius of convergence, and therefore we analyse the possibility of obtaining analytic functions of which these power series are asymptotic expansions by means of Borel re-summation. This process is complete for torus knots, and a partial answer is presented in the general case, which gives an upper bound on the growth of the coefficients of the Melvin-Morton expansion of the Coloured Jones Polynomial. In the Lorentz group case, this perturbative approach is proved to coincide with the algebraic and combinatorial approach for knot invariants defined out of the formal R-matrix and formal ribbon elements in the Quantum Lorentz Group, and its infinite dimensional unitary representations.

Acknowledgements

This work was financially supported by the programme *" PRAXIS-XXI"*, grant number SFRH/BD/1004/2000 of *Fundação para a Ciência e a Tecnologia (FCT)*, financed by the European Community fund *Quadro Comunitario de Apoio III*.

I would like to express my gratitude to my supervisor Dr John W. Barrett for all the support he gave me in the course of my PhD, for the inspiration he provided me with, and for his ability to fight against my pessimism.

Introduction

Since the advent of quantum groups, and in particular of the notion of a quantised universal enveloping algebra of a semisimple Lie algebra in the end of the eighties, their theory has been applied to the construction of link invariants. The main idea behind all approaches comes from the observation that, in the current terminology, they are ribbon Hopf algebras [RT], which implies that their category of finite dimensional representations is a ribbon category, with trivial associativity constraints. In the pioneering work of Freyd and Yetter, cf. [FY], it was observed that the (ribbon) tangles form a ribbon category which is universal in the class of all strict ribbon categories. This framework gives us a knot invariant for any ribbon Hopf algebra and any finite dimensional representation of it, as observed in the construction of Reshetikin and Turaev's functor defined in [RT].

A limitation of the constructions above is that they are not directly applicable to the case of infinite dimensional representations of ribbon Hopf algebras. This is because they involve taking traces or the use of coevaluation maps, which are difficult to define in the infinite dimensional context. However, one is forced to deal with knot invariants associated with infinite dimensional representations when considering invariants associated with unitary representations of non-compact Lie groups. This kind of representation appears in the context of $(2+1)$-quantum gravity and Chern-Simons theory with non-compact groups. See for example [W],[BC],[BNR], [GI], [NR] or [G]. It would thus be important to define $\mathbb{C}$-valued knot invariants associated with representations of this kind, natural observables for $(2+1)$-dimensional quantum gravity and Lorentzian Chern-Simons theory. The aim of the thesis, is to describe a possible path for doing this. We shall be mostly interested in the $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ cases.

The h-adic quantised universal enveloping algebras of semisimple Lie algebras are usually easier to deal with in the context of infinite dimensional representations. In this case there are various different variants of the construction of quantum Knot invariants. Some of them can be used in the infinite dimensional case, for example, Lawrence's Universal $U_h(\mathfrak{g})$ knot invariant or the Kontsevich Universal knot invariant. Roughly speaking, given a (complex semisimple) Lie algebra and an ad-invariant non-degenerate symmetric bilinear form on it, they will yield a knot invariant which takes values in the algebra of formal power series over the centre of $U(\mathfrak{g})$, the universal enveloping algebra of $\mathfrak{g}$. It is called the universal $U(\mathfrak{g})$-knot invariant. These power series define an analytic function from $\mathbb{C}$ to a completion of the universal enveloping algebra of $\mathfrak{g}$ when it is given the topology of convergence in its finite dimensional representations. One of the aims of this thesis is to consider also infinite dimensional representations of $\mathfrak{g}$.

The main idea behind the construction of non-compact group knot invariants is the following. Suppose we have a representation ρ of the Lie algebra $\mathfrak{g}$ in some vector space V, which we do not assume to be finite dimensional. We can always lift it to a representation, also denoted by ρ, of the enveloping algebra of $\mathfrak{g}$. In some cases it can happen that any element of the centre of $U(\mathfrak{g})$ acts in V as a multiple of the identity. Such representations thus define an algebra morphism from the centre of $U(\mathfrak{g})$ to $\mathbb{C}$, in other words: a central character of $U(\mathfrak{g})$. They are usually called representations which admit a central character. This type of $\mathfrak{g}$-module arises naturally in Lie algebra theory. Some examples would be the cyclic highest weight representations of a semisimple Lie algebra. Notice they are infinite dimensional if the weight is not integral dominant. It is a well established fact that the central characters of them exhaust all central characters of $U(\mathfrak{g})$ if $\mathfrak{g}$ is complex semisimple. Another context where representations which admit a central character appears is the context of unitary irreducible representations R of real Lie Groups G with Lie algebra $\mathfrak{g}$ in complex Hilbert spaces V. More precisely, it is possible to prove that the induced representation R^∞ of $U(\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})$ in the space of smooth vectors of V under the action of R admits a central character. It is called the infinitesimal character of R.

Any central character of $U(\mathfrak{g})$ can be used to evaluate the universal $U(\mathfrak{g})$ knot invariant. This will then yield a knot invariant with values in the algebra of formal power series over $\mathbb{C}$. For example if we use the central character of the highest weight representations of $\mathfrak{sl}(2,\mathbb{C})$, we obtain exactly the Melvin-Morton expansion of the Coloured Jones Polynomial, expansion which we called z-coloured Jones Polynomial. Obviously, one price we have to pay when we consider infinite dimensional representation of $\mathfrak{g}$ will then be, in general, the need to stick to representations of $\mathfrak{g}$ that admit a central character and to links with one component (knots). In this thesis, we propose to consider this kind of knot invariant for infinite dimensional irreducible unitary representations of $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$, and to analyse precisely its algebraic and convergence properties. Let us agree to call them non-compact knot invariants.

As mentioned before, in the semisimple Lie algebras context the central characters of the highest weight representations exhaust all central characters of $U(\mathfrak{g})$. Moreover the value of these central characters in a central element of $U(\mathfrak{g})$ depends polynomially on the weight and it is determined by its values on the weights that define finite dimensional representations. Therefore, in the power series level these non-compact invariants are analytic continuation of the usual quantum groups knot invariants. For example in the $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ case these knot invariants are thus obviously contained in the Melvin-Morton expansion of the Coloured Jones Polynomial. This exact relation is calculated in theorems 26 and 16. In particular the knot invariants obtained by admitting infinite dimensional representations are not more powerful than the already known ones. Therefore the main motivation for this thesis concerns the applications of these knot invariants to mathematical physics and geometric topology. It is unclear what happens in the non-semisimple Lie algebras context.

As we referred to before, the definition of $\mathbb{C}$-valued, that is numerical, knot invariants would be the most important for applications. So we want to say something about about the kind of power series that we obtain. This will be one of the main subjects of this thesis. A main result will be that for a large class of interesting unitary infinite dimensional representations of $SL(2,\mathbb{R})$ the associated series has a zero radius of convergence, at least in the case of torus knots. The same is true in the $SL(2,\mathbb{C})$ case. Notice that this does not happen in the case of finite dimensional representations, in which case there is no problem in defining numerical knot invariants. However, in the context of torus knots, they define Borel re-summable series. This means there is a natural way to find analytic functions of which these power series are asymptotic expansions; and also, that the uncertainty in the process of re-summation is reduced to a numerable, in this case finite, set of functions, differing by rapidly decreasing terms. It would be interesting to analyse what this uncertainty means geometrically. This re-summation realises an analytic extension of the coloured Jones polynomial of torus knots to complex spins, in the context of numerical knot invariants rather than only termwise in the power series, that is, of the actual values of the coloured Jones polynomial. In the general case of an arbitrary knot and a unitary representation of $SL(2,\mathbb{R})$ or $SL(2,\mathbb{C})$, the main result obtained (theorem 34) is that the series obtained are of Gevrey type 1. This ensures there is an upper bound for the divergence of the power series. This is a necessary condition for Borel summability and permits us to define a weaker process of re-summation up to exponentially decreasing functions. It is an open problem whether the process of Borel re-summation of the z-coloured Jones polynomial works for any knot.

In the Lorentz Group case, the program described until now relates with the work of Buffenoir and Roche, [BR1] and [BR2] on the representation theory of the quantum Lorentz group. We use the quantum Lorentz group defined by Woronowicz and Poddles in [PW]. This thesis contains a full description of the knot theory related to it in the representative example of balanced (simple) representations, as well as the precise relation with the previous perturbative framework. Let us be a bit more explicit. Suppose $\mathcal{A}$ is a Hopf algebra, its category of finite dimensional representations is therefore a compact monoidal category. Let q be a complex number not equal to 1 or -1. Suppose $\mathcal{A} = U_q(\mathfrak{g})$ is the Drinfeld-Jimbo algebra attached to the semisimple Lie algebra $\mathfrak{g}$. Even though $\mathcal{A}$ is not a ribbon Hopf algebra, it possesses a formal R-matrix and a formal ribbon element. These elements make sense when applied to finite dimensional representations of $\mathcal{A}$, and thus its category of finite dimensional representations is a ribbon category. This means we have a knot invariant attached to any finite dimensional representation of $\mathcal{A}$. This kind of knot invariant takes values in $\mathbb{C}$.

A similar situation happens in the case of the Quantum Lorentz Group $\mathcal{D}$, also denoted by $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$. Despite the fact $\mathcal{D}$ is not a Drinfeld Jimbo algebra, its structure of a quantum double, namely $\mathcal{D} = \mathcal{D}(U_q(\mathfrak{su}(2)), \mathrm{Pol}(SU_q(2))$ with $q \in (0,1)$, makes possible the definition of a formal R-matrix on it. Also, it is possible to define a heuristic ribbon element. A fact observed in [BR1] is that we can describe the action of the formal R-matrix of the Quantum Lorentz Group in a class of infinite dimensional representations of it. For this reason, it is natural to ask whether there exists a knot theory attached to the infinite representations of $\mathcal{D}$. This would generalise the work of Barrett and Crane in [BC], extending their spin foam model to the quantum case in which the evaluation of (infinite dimensional) Lorentzian spin networks would be sensitive to knotting (This extension has been done in [NR]). We shall see the answer is affirmative at least in the perturbative level. As we mentioned before, since we are working with infinite dimensional representations the general formulation of Reshetikin and Turaev for constructing knot invariants cannot be directly applied. We use now a slightly different way of thinking to get around this problem. It is possible, given a knot diagram, or to be more precise a connected $(1,1)$-tangle diagram, to make a heuristic evaluation of the Reshetikin-Turaev functor on it. This yields an infinite series for any knot diagram. This method was also elucidated in [NR]. Unfortunately, at least for unitary infinite dimensional representations, this infinite series do not seem to be convergent , which tells us that the process of Borel re-summation is perhaps more powerful. However, they converge h-adically for $q = \exp(h/2)$, since the expansions of their terms as power series in h starts in increasing degree of h^n. Therefore these evaluations do define $\mathbb{C}[[h]]$-valued knot invariants. A main result of this thesis is that it is possible to choose an ad-invariant inner product in the Lie algebra of the Lorentz group such that the knot invariants coming out of the infinite dimensional representations of the Lorentz Group in the framework of the Kontsevich Integral are exactly these ones. This bilinear form is first conjectured in chapter 2 and then proved to be the good one in chapter 7 (Theorem 136).

A resume of this thesis for the expert

First of all a note about proofs. I included a large amount of background material in this thesis, insisting in a pedagogical, simple and self-contained exposition. I decided to provide proofs of known results when they include some important ideas which are used afterwards, for example Theorem 5; if the result will have a major important for the sequel, for example the relation between framed and unframed Kontsevich Integral in 6.1.3; and it is difficult to find a proof of it in the literature, for example for theorem 2; or the proof is too complicated for the reader to understand it without reading the rest of the article for example in 7.3.3, or it is not clear that it is correct in the way it is proved.

Chapter one

This contains mainly background material, explaining the main results needed and especially the philosophy of this thesis. It is tells the reader mainly about the way we can define knot invariants from the Kontsevich Integral and weight systems, together with their relation with the Coloured Jones Polynomial and the Melvin Morton Expansion of it.

Chapter Two

It uses the framework of the previous chapter to define knot invariants from the infinite dimensional representations of the Lorentz Group. The main result of this chapter concerns the way we can obtain them out of the Coloured Jones Polynomial.

Chapter Three

This is one of the main chapters of this thesis. It presents the general framework for dealing with knot invariants defined from infinite dimensional representations of Lie Groups, and the way they can always be obtained from the usual quantum groups knot invariants in the semisimple case.

Chapter Four

This is, definitely, the most important chapter of this thesis. It contains a proof that the Melvin Morton expansion of the Coloured Jones Polynomial defines power series with a zero radius of convergence for torus knots. It contains background material on Borel re-summations and a proof that the $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ invariants are

Borel re-summable for torus knots. A weaker result of re-summability is stated in the general case (Theorem 34). It contains a list of open problems.

Chapter Five

It is a mixture of background material and some technical proofs of theorems stated in chapters 3 and 4, as well as a condensed proof of Theorem 34, and a simple proof of it for the Figure of Eight Knot

Chapter Six

It is a very technical and heavy chapter. It presents the Kontsevich Integral, and gives bounds for its coefficients as well as bounds for the evaluation of chord diagrams, probably interesting by themselves. The main result is a full proof of Theorem 34, an upper bound in the growth of the coefficients of the Melvin-Morton Expansion of the Coloured Jones Polynomial.

Chapter Seven

The subject is slightly different to the rest of this thesis. The main aim is to prove the exact relation between the previous perturbative framework and the knot invariants defined from the unitary representation of the quantum Lorentz Group. It contains background material on the Quantum Lorentz Group, and a study of its representation theory, made as parallel as possible with the classical case described in Chapter The main result of this chapter is a precise proof that we can define $\mathbb{C}[[h]]$-knot invariants from infinite dimensional representations of the Lorentz group and how they relate with the previous perturbative approach. Some attempts to analyse the convergence or divergence of the associated series of complex numbers are made.

Important References

Out out the very big list of references included, I would like to select some very important ones, necessary to follow this thesis. These are: [BR2],[C],[CD], [CV], [LM1], [M] and [K]. This thesis is an expanded version of [FM1] and [FM2].

I started developing my interest in Mathematics in the very early stages of my life with the best teacher I ever had: My Father. This thesis is dedicated to him.

1 Preliminaries

1.1 Chord diagrams

We recall the definition of the algebra of chord diagrams, which is the target space for the Kontsevich Universal Knot Invariant. For more details see for example [BN] or [K]. A chord diagram is a finite set $w = \{c_1, ..., c_n\}$ of cardinality 2 nonintersecting subsets of the oriented circle, modulo orientation preserving homeomorphisms. The subsets c_k are called chords and are supposed to be pairwise disjoint. We usually specify a chord diagram by drawing it as in figure 1. In all the pictures we assume the circle oriented counterclockwise.

For each $n \geq 2$, let V_n be the free $\mathbb{C}$ vector space on the set of all chord diagrams with n chords. That is the set of formal finite linear combinations $w = \sum_i \lambda_i w_i$, where $\lambda_i \in \mathbb{C}$ and w_i is a chord diagram with i chords for any i. Consider the sub vector space $4T_n$ of V_n which is the subspace generated by all linear combinations of chord diagrams of the form displayed in figure 2. The 3 intervals considered in the circle can appear in an arbitrary order in S^1. Define for each $n \in \mathbb{N}_0 = \{0, 1, 2, ..\}$,the vector space $\mathcal{A}_n = V_n/4T_n$. We consider $\mathcal{A}_0 = V_0$ and $\mathcal{A}_1 = V_1$.

For any pair $m, n \in \mathbb{N}_0$, there exists a bilinear map $\# : \mathcal{A}_n \otimes \mathcal{A}_m \to A_{m+n}$, called the connected sum product. As its name says, it is performed by doing the connected sum of chord diagrams as in figure 3.

Obviously the product is not well defined in $V_m \otimes V_n$ since it depends on the points in which we break the circles. However, the connected sum product makes sense in $\mathcal{A}_m \otimes \mathcal{A}_n$, for we are considering the 4-term relations. We explain why it is so in 5.1. It is an associative and commutative product, and it has a unit: the chord diagram

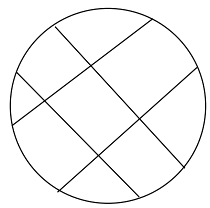

Figure 1: A chord diagram with 4 chords

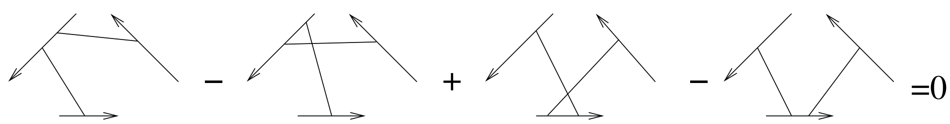

Figure 2: 4 Term relations

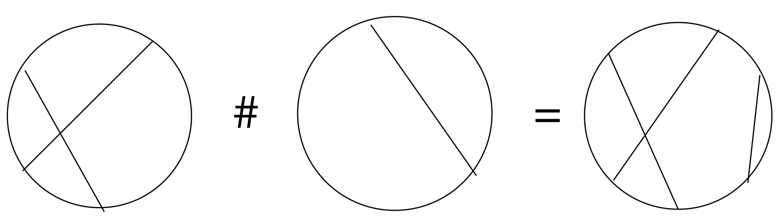

Figure 3: Connected sum product

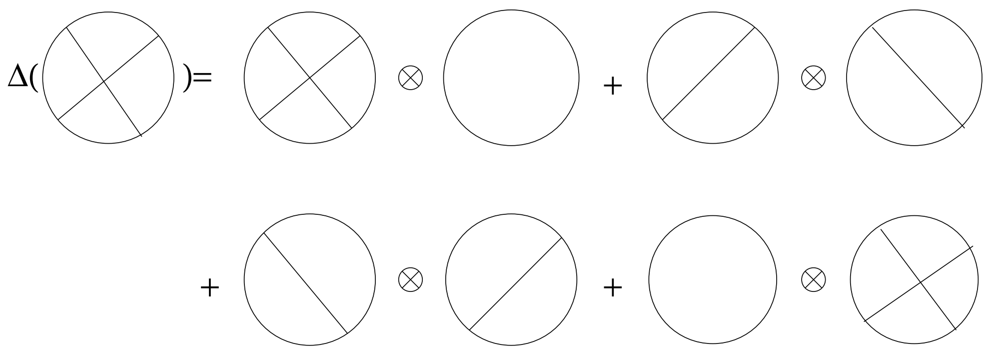

Figure 4: Coproduct maps

without any chord. For more details see [BN].

The vector space $\mathcal{A}_m \otimes \mathcal{A}_m$ is mapped via the connected sum product to $\mathcal{A}_{m+n}$. Therefore the direct sum $\mathcal{A}_{\text{fin}} = \bigoplus_{n\in\mathbb{N}_0} \mathcal{A}_n$ has a commutative and associative graded algebra structure, where the grading coefficient of a chord diagram is given by its number of chords. This permits us to conclude that the vector space

$$\mathcal{A} = \prod_{n\in\mathbb{N}_0} \mathcal{A}_n$$

has a structure of abelian algebra over the field of complex numbers. It is the graded completion of $\mathcal{A}_{\text{fin}}$. Call it the algebra of chord diagrams. The algebra $\mathcal{A}$ is the target space for the Kontsevich Universal Knot Invariant. There exist also coproduct maps $\Delta : \mathcal{A}_m \to \bigoplus_{k+l=m} A_k \otimes \mathcal{A}_l$ which have the form of figure 4 on chord diagrams.

They extend to a linear map $\Delta : \mathcal{A} \to \mathcal{A}\hat{\otimes}\mathcal{A}$. Here $\mathcal{A}\hat{\otimes}\mathcal{A}$ is the vector space

$$\prod_{m\in\mathbb{N}_0} \bigoplus_{k+l=m} \mathcal{A}_k \otimes \mathcal{A}_l.$$

Notice that $\mathcal{A} \otimes \mathcal{A}$ is a proper sub vector space of $\mathcal{A}\hat{\otimes}\mathcal{A}$.

An element $w \in \mathcal{A}$ is called group like if $\Delta(w) = w\hat{\otimes}w$. That is, if writing $w = \sum_{n\in\mathbb{N}_0} w_n$ with $n \in \mathcal{A}_n, \forall n \in \mathbb{N}_0$ we have

$$\Delta(w_n) = \sum_{l+k=n} w_k \otimes w_l.$$

For example, $\exp(\ominus)$ is a group like element. Here $\ominus$ is the unique chord diagram with only one chord. This is a trivial consequence of the fact $\Delta(\ominus) = \ominus \otimes 1 + 1 \otimes \ominus$. We have put 1 for the chord diagram without chords.

1.2 The Kontsevich Universal Knot Invariant

We skip for a moment the definition of the (framed) Kontsevich Universal Knot invariant $\mathcal{Z}$, for which we refer for example to [K], [LM1], [Wi1] and 6.1. For the unframed version see [CD] and [BN], the classical reference. The sources [LM2] and [Wi1] unify the two theories in a nice way, as we describe in 6.1.3. We take the normalisation of the Kontsevich Universal Knot Invariant for which the value of the unknot is the wheels element Ω of [LNT]. That is $\mathcal{Z}(O) = \mathbf{Z}(\infty)$, cf [BN] pp 447, or 6.1. This is a different normalisation of the one used in [BN]. We now gather the properties of the Kontsevich Universal Knot Invariant which we are going to use in the sequel. Notice that the product we consider in $\mathcal{A}$, the algebra of chord diagrams, is the connected sum product $\#$.

Theorem 1 *There exists a (oriented and framed) knot invariant $K \mapsto \mathcal{Z}(K)$, where $\mathcal{Z}(K)$ is in the algebra $\mathcal{A}$ of chord diagrams. Given a framed knot K, $\mathcal{Z}(K)$ satisfies:*

1. *$\mathcal{Z}(K)$ is grouplike, cf [BN]*

2. *If K^f is obtained from $\mathcal{K}$ by changing its framing by a factor of* 1 *then* $\mathcal{Z}(K^f) = \mathcal{Z}(K)\exp(-\ominus)$*, cf [LM1].*

3. *If K^* is the mirror image of K, and writing $\mathcal{Z}(K) = \sum_{n\in\mathbb{N}_0} w_n$ with $\omega_n \in \mathcal{A}_n, \forall n \in \mathbb{N}_0$ we have $\mathcal{Z}(K^*) = \sum_{n\in\mathbb{N}_0}(-1)^n w_n$, cf [CD].*

4. *If K^- is the knot obtained from K by reversing the orientation of it then $\mathcal{Z}(K^-) = \sum_{n\in\mathbb{N}_0} S(w_n)$. Here $S : \mathcal{A}_n \to \mathcal{A}_n$ is the map that reverses the orientation of each chord diagram, cf [CD].*

5. *$\mathcal{Z}(K\#L) = \mathcal{Z}(K)\mathcal{Z}(L)\mathcal{Z}(O)^{-1}$, where $K\#L$ denotes the connected sum of the knots K and L and O is the unknot.*

Suppose we are given a family of linear maps (weights) $W_n : \mathcal{A}_n \to \mathbb{C}, n \in \mathbb{N}_0$. A knot invariant whose value on each knot is a formal power series with coefficients in $\mathbb{C}$ is called canonical if it has the form

$$K \mapsto \sum_{n\in\mathbb{N}_0} W_n(w_n)h^n.$$

As usual we write $\mathcal{Z}(K) = \sum_{n\in\mathbb{N}_0} w_n$ with $w_n \in \mathcal{A}_n, \forall n \in \mathbb{N}_0$.

1.3 Infinitesimal R-matrices

Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its universal enveloping algebra. Notice $U(\mathfrak{g})$ is generated as an algebra by 1 and $\mathfrak{g}$. It has a unique Hopf algebra structure for which:

1. $\Delta(X) = X \otimes 1 + 1 \otimes X, \forall X \in \mathfrak{g}$

2. $\varepsilon(X) = 0, \forall X \in \mathfrak{g}$

3. $S(X) = -X, \forall X \in \mathfrak{g}$

Let $\mathfrak{g}$ be a Lie algebra over the field $\mathbb{C}$[1]. An infinitesimal R-matrix of $\mathfrak{g}$ is a symmetric tensor $t \in \mathfrak{g} \otimes \mathfrak{g}$ such that $[\Delta(X), t] = 0, \forall X \in \mathfrak{g}$. The commutator is taken in $U(\mathfrak{g}) \otimes U(\mathfrak{g})$.

Suppose we are given an infinitesimal R-matrix t. Write $t = \sum_i a_i \otimes b_i$. We have $\sum_j [\Delta(a_j), t] \otimes b_j = 0$, thus:

$$\sum_{i,j} a_j a_i \otimes b_i \otimes b_j - a_i a_j \otimes b_i \otimes b_j + a_i \otimes a_j b_i \otimes b_j - a_i \otimes b_i a_j \otimes b_j = 0, \qquad (1)$$

which resembles the $4T$ relations considered previously. Given a chord diagram w and an infinitesimal R-matrix $t = \sum_i a_i \otimes b_i$ it is natural thus to construct an element $\phi_t(w)$ of $U(\mathfrak{g})$ in the following fashion: Start in an arbitrary point of the circle and go around it in the direction of its orientation. Order the chords of w by the order with which you pass them as in figure 5. Each chord has thus an initial and an end point. Then go around the circle again and write (from the right to the left) a_{i_k} or b_{i_k} depending on whether you got to the initial or final point of the k^{th} chord. Then sum over all the i_k's. For example for the chord diagram of figure 5 the element $\phi_t(w)$ is:

$$\sum_{i_1, i_2, i_3} b_{i_2} b_{i_3} b_{i_1} a_{i_3} a_{i_2} a_{i_1}.$$

See [K] or [CV] for more details. It is possible to prove (we are going to do it in 5.1)

[1] We could obviously consider any field, however it needs to be a subfield of $\mathbb{C}$ so that the evaluation of chord diagrams make sense. Notice the the Kontsevich Universal knot invariant has values in the algebra of chord diagrams with coefficients in $\mathbb{Q}$

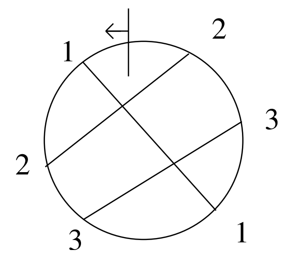

Figure 5: Enumerating the Chords of a Chord Diagram

that $\phi_t(w)$ is well defined as an element of $U(\mathfrak{g})$, that is it does not depend on the starting point in the circle. Moreover:

Theorem 2 *Let $\mathfrak{g}$ be a Lie algebra and $t \in \mathfrak{g} \otimes \mathfrak{g}$ be an infinitesimal R-matrix. The linear map $\phi_t : V_n \to U(\mathfrak{g})$ satisfies the 4T relations, therefore it descends to a linear map $\phi_t : \mathcal{A}_n \to U(\mathfrak{g})$. Moreover:*

1. *The image of ϕ_t is contained in $\mathcal{C}(U(\mathfrak{g}))$, the centre of $U(\mathfrak{g})$.*
2. *The degree of $\phi_t(w)$ in $U(\mathfrak{g})$ with respect to the natural filtration of $U(\mathfrak{g})$ is not bigger than twice the number of chords of w.*
3. *Given $w \in \mathcal{A}_m$ and $w' \in \mathcal{A}_n$ we have $\phi_t(w\#w') = \phi_t(w)\phi_t(w')$*
4. *Consider the map $\phi_{t,h} : \mathcal{A} \to \mathcal{C}(U(\mathfrak{g}))[[h]]$ such that if $w = \sum_{n \in \mathbb{N}_0} w_n$ with $w_n \in \mathcal{A}_n$ for each $n \in \mathbb{N}_0$ we have*

$$\phi_{t,h} = \sum_{n \in \mathbb{N}_0} \phi_t(w_n) h^n.$$

Then $\phi_{t,h}$ is a $\mathbb{C}$-algebra morphism.

Usually we put ϕ_t instead of $\phi_{t,h}$ to simplify the notation. This is a well known result. However I could not find a complete proof of it in the literature. A large part of this thesis depends on it, therefore we will give a proof of the difficult part of it in 5.1.

Obviously, depending on the Lie algebra and the infinitesimal R-matrix we choose, the evaluation of ϕ_t will satisfy some additional relations to the 4-term relations. We will give a description of this for the $\mathfrak{sl}(2,\mathbb{C})$ case in 5.2, following [CV]. As we will see in the next section, apart from scaling, there exists only one infinitesimal R-matrix in $\mathfrak{sl}(2,\mathbb{C})$, and in general in any simple Lie algebra.

1.3.1 Constructing infinitesimal R-matrices

There exists a standard way to construct infinitesimal R-matrices in a Lie algebra $\mathfrak{g}$. Suppose we are given a $\mathfrak{g}$-invariant, non degenerate, symmetric bilinear form $<,>$ in $\mathfrak{g}$. Here $\mathfrak{g}$-invariance means that we have $<[X,Y],Z> + <Y,[X,Z]>= 0, \forall X,Y,Z \in \mathfrak{g}$. If $\mathfrak{g}$ is semisimple the Cartan-Killing form verifies the properties above. Take a basis $\{X_i\}$ of $\mathfrak{g}$ and let $\{X^i\}$ be the dual basis of $\mathfrak{g}^*$. Then it is easy to show that for any $\lambda \in \mathbb{C}$ the tensor $t = \lambda \sum_i X_i \otimes X^i$ is an infinitesimal R-matrix of $\mathfrak{g}$. We are identifying $\mathfrak{g}^*$ with $\mathfrak{g}$ using the nondegenerate bilinear form $<,>$.

Suppose $\mathfrak{g}$ is a semisimple Lie algebra and let $t = \sum_i a_i \otimes b_i$ be an infinitesimal R-matrix in $\mathfrak{g}$. Let also $<,>$ denote the Cartan-Killing form on $\mathfrak{g}$. Then the map $\mathfrak{g} \to \mathfrak{g}$ such that $X \mapsto \sum_i <X, a_i> b_i$ is an intertwiner of $\mathfrak{g}$ with respect to its adjoint representation. Therefore if $\mathfrak{g}$ is simple it is a multiple λ of the identity. This permits us to conclude that $t = \lambda X_i \otimes X^i$.

Let us now look at the case when $\mathfrak{g}$ is semisimple. Then $\mathfrak{g}$ has a unique decomposition of the form $\mathfrak{g} \cong \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_n$, where each $\mathfrak{g}_i$ is a simple Lie algebra. The Cartan-Killing form in each $\mathfrak{g}_i$ will yield an infinitesimal R-matrix t_i in each $\mathfrak{g}_i$. Obviously each linear combination $t = \lambda_1 t_1 + ... + \lambda_n t_n$ is an infinitesimal R-matrix for $\mathfrak{g}$. An argument similar to the one before proves that any infinitesimal R-matrix in $\mathfrak{g}$ needs to be of this form.

It should be said that in the case in which an infinitesimal R-matrix in a Lie algebra $\mathfrak{g}$ comes from a non-degenerate, symmetric and $\mathfrak{g}$-invariant bilinear form then our construction of central elements yields the same result as [BN], cf [CV].

1.3.2 Universal $U(\mathfrak{g})$-knot invariants

Recall that the Kontsevich integral is a sum of the form $\mathcal{Z}(K) = \sum_{n \in \mathbb{N}_0} w_n$ with $w_n \in \mathcal{A}_n, \forall n \in \mathbb{N}_0$. Let $\mathfrak{g}$ be a Lie algebra and t an infinitesimal R-matrix of $\mathfrak{g}$. We can consider the composition $\phi_t \circ \mathcal{Z}$. This will yield a knot invariant with values in the algebra of formal power series over the centre $\mathcal{C}(U(\mathfrak{g}))$ of $U(\mathfrak{g})$. Therefore:

Theorem 3 *Let $\mathfrak{g}$ be Lie algebra and t an infinitesimal R-matrix in $\mathfrak{g}$ There exists a framed knot invariant $\mathcal{Z}_t = (\phi_t \circ \mathcal{Z})$. It has the form:*

$$\mathcal{Z}_t = (\phi_t \circ \mathcal{Z}) : K \mapsto \sum_{n=0}^{+\infty} (\phi_t \circ \mathcal{Z}_n)(K) h^n, \tag{2}$$

where $(\phi_t \circ \mathcal{Z}_n)(K) \in \mathcal{C}(U(\mathfrak{g})), n = 0, 1...$

If $\mathfrak{g}$ is simple then according to the results of 1.3.1 there exists essentially only one infinitesimal R-matrix t. In this case $(\phi_t \circ \mathcal{Z})$ is called the universal $U(\mathfrak{g})$ knot invariant.

If $\mathfrak{g}$ is complex semisimple and t is the infinitesimal R-matrix coming from the Cartan-Killing form in $\mathfrak{g}$, then $(\phi_t \circ \mathcal{Z})(K)$ defines an analytic function from $\mathbb{C}$ to the completion of $U(\mathfrak{g})$, where $U(\mathfrak{g})$ is given the topology of convergence in its finite dimensional representations, cf. [PS]. In other words, for any finite dimensional representation ρ of $\mathfrak{g}$ in V, the power series $\rho(\phi_t \circ \mathcal{Z})(K)$ converges to a linear operator $V \to V$, in fact to a multiple of the identity if V is irreducible. The usual quantum group knot invariants are obtained by taking the trace of these operators. We will go back to these issues later in 3.3 for the $\mathfrak{sl}(2,\mathbb{C})$ case. This thesis aims mostly to consider the case in which we admit infinite dimensional representations.

1.3.3 $\mathbb{C}[[h]]$-valued knot invariants

Let $\mathfrak{g}$ be a Lie algebra and t be an infinitesimal R-matrix in $\mathfrak{g}$. Suppose are given a morphism $f : \mathcal{C}(U(\mathfrak{g})) \to \mathbb{C}$. Morphisms f like this are usually called central characters of $\mathfrak{g}$. Composing f with $\mathcal{Z}_t = (\phi_t \circ \mathcal{Z})$ yields a canonical knot invariant (see the end of 1.2) $f \circ \phi_t \circ \mathcal{Z}$.

Obviously any map $f : \mathcal{C}(U(\mathfrak{g})) \to \mathbb{C}$ will define a knot invariant in the same fashion. However, if f is a morphism then $f \circ \phi_t : \mathcal{A} \mapsto \mathbb{C}[[h]]$ is also a morphism of algebras thus the properties 2 and 5 of theorem 1 will translate in the obvious way to $f \circ \phi_t \circ \mathcal{Z}$.

It is not difficult to examine the conditions whereby this kind of knot invariants are unframed. Let $t = \sum_i a_i \otimes b_i$ be an infinitesimal R matrix in a Lie algebra. Define $C_t = \sum_i a_i b_i = \phi_t(\ominus)$. It is a central element of the universal enveloping algebra of $\mathfrak{g}$. Call it the quadratic central element associated with t. The infinitesimal R-matrix t can be recovered from C_t by the formula

$$t = \frac{\Delta(C_t) - 1 \otimes C_t - C_t \otimes 1}{2}.$$

A morphism $f : \mathcal{C}(U(\mathfrak{g})) \to \mathbb{C}$ is said to be t-unframed if $f(C_t) = 0$. From theorem 1, 2 and theorem 2, 3 it is straightforward to conclude that:

Theorem 4 *Let $\mathfrak{g}$ be a Lie algebra with an infinitesimal R-matrix t. Consider also a morphism f from the centre of $U(\mathfrak{g})$ to $\mathbb{C}$. Then the knot invariant $f \circ \mathcal{Z}_t$ is unframed if and only if the morphism f is t-unframed.*

Notice the Kontsevich integral of each knot is invertible in $\mathcal{A}$. This is because the term $w_0 \in \mathcal{A}_0$ is the unit of $\mathcal{A}$. We need this fact to prove the last theorem.

1.3.4 A factorisation theorem

Suppose the Lie algebra $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is the direct sum of two Lie algebras. If t_1 and t_2 are infinitesimal R-matrices in $\mathfrak{g}_1$ and $\mathfrak{g}_2$ then $t = t_1 + t_2$ is also an infinitesimal R-matrix in $\mathfrak{g}$. Moreover given a chord diagram w we have the following identity: cf [BN]

$$\phi_t(w) = (\phi_{t_1} \otimes \phi_{t_2})\Delta(w).$$

It is trivial to conclude this from the definition of Δ. We are obviously considering the standard isomorphism $U(\mathfrak{g}) \cong U(\mathfrak{g}_1) \otimes U(\mathfrak{g}_2)$ such that $(X, Y) \mapsto X \otimes 1 + 1 \otimes Y$ for $(X, Y) \in \mathfrak{g}$.

If we are given two algebra morphisms $f_i : \mathcal{C}(U(\mathfrak{g}_i)) \to \mathbb{C}, i = 1, 2$, then $f = f_1 \otimes f_1$ is an algebra morphism $\mathcal{C}(U(\mathfrak{g})) \cong \mathcal{C}(U(\mathfrak{g}_i)) \otimes \mathcal{C}(U(\mathfrak{g}_i) \to \mathbb{C}$. We can thus consider the knot invariant $f \circ \mathcal{Z}_t$. It expresses in a simple form in terms of $f_i \circ \mathcal{Z}_{t_i}, i = 1, 2$. In fact:

Theorem 5 *Given any (oriented and framed) knot K we have:*

$$(f \circ \mathcal{Z}_t)(K) = (f_1 \circ \mathcal{Z}_{t_1})(K) \times (f_2 \circ \mathcal{Z}_{t_2})(K),$$

as formal power series

Proof.

Let K be a knot, write $\mathcal{Z}(K) = \sum_{n \in \mathbb{N}_0} w_n$ with $w_n \in \mathcal{A}_n, \forall n \in \mathbb{N}_0$. We have:

$$\begin{aligned}
(f \circ \mathcal{Z}_t)(K) &= \sum_{n \in \mathbb{N}_0} (f \circ \phi_t)(w_n) h^n \\
&= \sum_{n \in \mathbb{N}_0} (f_1 \otimes f_2) \circ (\phi_{t_1} \otimes \phi_{t_2})(\Delta(w_n)) h^n \\
&= \sum_{n \in \mathbb{N}_0} \sum_{k+l=n} (f_1 \otimes f_2) \circ (\phi_{t_1} \otimes \phi_{t_2})(w_k \otimes w_l) h^n \\
&= \sum_{n \in \mathbb{N}_0} \sum_{k+l=n} \left[(f_1 \circ \phi_{t_1})(w_k)\right] \left[(f_2 \circ \phi_{t_2})(w_l)\right] h^{k+l} \\
&= (f_1 \circ \mathcal{Z}_{t_1})(K) \times (f_2 \circ \mathcal{Z}_{t_2})(K).
\end{aligned}$$

■

This result and proof appears in [GN]

1.4 The Coloured Jones Polynomial

1.4.1 The algebra $U_h(\mathfrak{sl}(2, \mathbb{C}))$

The coloured Jones Polynomial is constructed out of the finite dimensional representations of the h-adic Hopf algebra $U_h(\mathfrak{sl}(2, \mathbb{C}))$, the quantised universal enveloping algebra of $\mathfrak{sl}(2, \mathbb{C})$. We will follow the convention of [H]. The algebra $U_h(\mathfrak{sl}(2, \mathbb{C}))$ is the complete $\mathbb{C}[[h]]$-algebra topologically generated by H , E and F and relations:

$$HE = E(H+2), HF = F(H-2), (EF - FE) = \frac{\sinh(hH/2)}{\sinh(h/2)}.$$

Let $q = \exp(h), v = \exp(h/2)$ and $K = \exp(hH/2)$. Set $[n] = [n]_h = \frac{v^n - v^{-n}}{v - v^{-1}}$ and $[n]_h! = [1][2]...[n]$. The algebra $U_h(\mathfrak{sl}(2,\mathbb{C}))$ can be given a structure of ribbon quasi Hopf algebra with R-matrix

$$R = v^{\frac{1}{2}H\otimes H} \sum_{n=0}^{+\infty} \frac{v^{\frac{n(n+1)}{2}}}{[n]!}(v - v^{-1})^n E^n \otimes F^n,$$

and group like element $G = K$. Therefore, the ribbon element θ is

$$\theta = K^{-1} \sum_S (t_i) s_i,$$

where $R = \sum_i s_i \otimes t_i$. See [RT], for example for the definition of ribbon Hopf algebras.

The theory of *finite dimensional* representations of $U_h(\mathfrak{sl}(2,\mathbb{C}))$ is similar to the representation theory of $\mathfrak{sl}(2,\mathbb{C})$. A complete set of indecomposable representations of $U_h(\mathfrak{sl}(2,\mathbb{C}))$ made out of the representations of spin $\alpha \in \frac{1}{2}\mathbb{N}_0$. These are representations in the free $\mathbb{C}[[h]]$-module with basis $\{v_0, ..., v_{2a}\}$ and actions:

$$Hv_i = (n-2i)v_i, \quad Ev_i = [n+1-i]v_{i-1}, \quad Fv_i = [i+1]v_{i+1}.$$

Alternatively, we can see it as a representation in $\overset{\alpha}{V}$, the free $C[[h]]$ module with basis $\{\overset{\alpha}{v}_{-\alpha}, ..., \overset{\alpha}{v}_a\}$, considering this actions of generators of $U_h(\mathfrak{sl}(2,\mathbb{C}))$:

$$\begin{aligned}
H\overset{\alpha}{v}_{i_\alpha} &= 2a_i \overset{\alpha}{v}_{i_\alpha},\\
E\overset{\alpha}{v}_{i_\alpha} &= v^{-\frac{1}{2}} v^{i_\alpha} \sqrt{[\alpha + i_\alpha + 1][\alpha - i_\alpha]}\,\overset{\alpha}{v}_{i_\alpha + 1},\\
F\overset{\alpha}{v}_{i_\alpha} &= v^{+\frac{1}{2}} v^{-i_\alpha} \sqrt{[\alpha - i_\alpha + 1][\alpha + i_\alpha]}\,\overset{\alpha}{v}_{i_\alpha - 1}.
\end{aligned}$$

The possibility of writing the representations in this way will be important later. Notice that the terms under the square roots always define holomorphic functions in a neighbourhood of zero, thus elements of $\mathbb{C}[[h]]$. These are all the indecomposable finite dimensional representations of $U_h(\mathfrak{sl}(2,\mathbb{C}))$. See [KS] or [CP].

1.4.2 Ribbon Hopf Algebras, knot invariants and the Coloured Jones Polynomial

Recall that any finite dimensional representation V of a ribbon Hopf algebra $\mathcal{A}$ always define a framed knot invariant I. Let us say what our conventions are. See [K],[RT] or [CP] for more details. Let $R = \sum_i s_i \otimes t_i$ be the R-matrix of $\mathcal{A}$ and θ its ribbon element. Recall the group like element G of $\mathcal{A}$ is $G = u\theta^{-1}$ where $u = \sum_i S(t_i)s_i$

Let $\mathcal{A}$ be a ribbon Hopf algebra with an R-matrix $R = \sum_i s_i \otimes t_i$, and group like element G. Let V be a finite dimensional representation of $\mathcal{A}$. Choose a basis $\{v_i\}$ of V and let $\{v^i\}$ be its dual basis of V^*. Notice V^* is also a representation of $\mathcal{A}$ under the rule $af(v) = f(S(a)v), v \in V, f \in V^*, a \in \mathcal{A}$. If K is a framed knot then $I(K)$ is calculated from a regular projection of K under the rules:

$$\mathcal{F}(\overset{\rightarrow}{\cup}) = 1 \mapsto \sum_i v_i \otimes v^i,$$

$$\mathcal{F}(\overset{\leftarrow}{\cup}) = 1 \mapsto \sum_i v^i \otimes G^{-1}v_i,$$

$$\mathcal{F}(\overset{\rightarrow}{\cap}) = f \otimes v \mapsto f(v),$$

$$\mathcal{F}(\overset{\leftarrow}{\cap}) = v \otimes f \mapsto f(Gv),$$

$$\mathcal{F}(X_+) = v \otimes w \mapsto \sum_i t_i w \otimes s_i v,$$

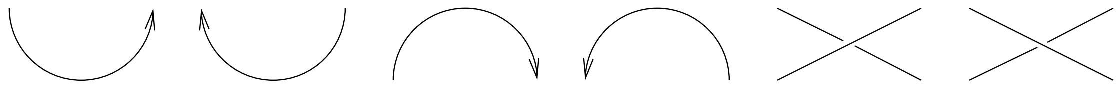

Figure 6: Knot Generators $\overrightarrow{\cup}$, $\overleftarrow{\cup}$, $\overrightarrow{\cap}$, $\overleftarrow{\cap}$, X_+ and X_-

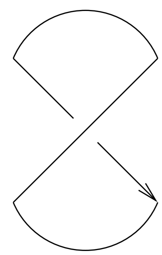

Figure 7: 1-framed unknot

$$\mathcal{F}(X_-) = w \otimes v \mapsto \sum_i \overline{s_i} w \otimes \overline{t_i} v,$$

where $R = \sum_i s_i \otimes t_i$ and $R^{-1} = \sum_i \overline{s_i} \otimes \overline{t_i}$

For example the evaluation for the 1-framed unknot in figure 7 is:

$$1 \mapsto \sum_i v^i \otimes G^{-1} v_i \mapsto \sum_{i,j} t_j G^{-1} v_i \otimes s_j v^i \mapsto \sum_{i,j} s_j v^i (G t_j G^{-1} v_i) =$$

$$\sum_{i,j} v^i (S(s_j) G t_j G^{-1} v_i) = \mathrm{tr}\{v \mapsto \sum_j v^i (S(s_j) G t_j G^{-1} v\} = \mathrm{tr}\{v \mapsto \theta u^{-1} v\}$$

$$= \mathrm{tr}\{v \mapsto \theta G^{-1} v\}.$$

Definition 6 *The coloured Jones polynomial* J^α*, where* $\alpha \in \frac{1}{2}\mathbb{N}_0$ *is, by definition the framed knot invariant made out of the the ribbon Hopf algebra* $U_h(\mathfrak{sl}(2, \mathbb{C}))$ *and the representation of spin* α *of it*

Therefore the coloured Jones polynomial is a knot invariant with values in $\mathbb{C}[[h]]$. In fact if we look at 1.4.1 it is trivial to conclude that $J^\alpha(K)$ is always a Laurent polynomial in $\exp(h/4) = v^{\frac{1}{2}}$.

1.4.3 The coloured Jones polynomial and central characters

Let $\mathfrak{g}$ be a simple Lie algebra and ρ a representation of $\mathfrak{g}$ in the vector space V. Then ρ is said to admit a central character if every element of $\mathcal{C}(U(\mathfrak{g}))$ acts on V as a multiple of the identity. In this case there exists an algebra morphism $\lambda_\rho : \mathcal{C}(U(\mathfrak{g})) \to \mathbb{C}$ such that $\rho(a)(v) = \lambda_\rho(a)v, \forall a \in \mathcal{C}(U(\mathfrak{g})), v \in V$. The algebra morphism λ_ρ is called the central character of the representation ρ. In particular, if $\mathfrak{g}$ is a Lie algebra with an infinitesimal R-matrix t then given any representation ρ of $\mathfrak{g}$ with a central character, we can construct the knot invariant $(\lambda_\rho \circ \phi_t \circ \mathcal{Z})$, as in 1.3.3. It has values in the algebra $\mathbb{C}[[h]]$ of formal power series over $\mathbb{C}$.

Recall that the coloured Jones polynomial J^α, where $\alpha \in \frac{1}{2}\mathbb{N}_0$ is, by definition the framed knot invariant made out of the the ribbon Hopf algebra $U_h(\mathfrak{sl}(2,\mathbb{C}))$ and the representation of spin α of it. It is therefore a framed knot invariant with values in $\mathbb{C}[[h]]$. However, apart from normalisation, the coloured Jones polynomial is a particular example of this construction. Let t be the infinitesimal R-matrix of $\mathfrak{sl}(2,\mathbb{C})$ corresponding to the bilinear form in it which is minus the Cartan-Killing form. Notice a nice form for t:

$$t = -\frac{1}{4}(\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z),$$

where $\sigma_X, \sigma_Y, \sigma_Z$ are the Pauli Matrices. See 2.1. Another expression for t appears in 1.4.6. Consider for any $\alpha \in \frac{1}{2}\mathbb{N}_0$ the representation $\overset{\alpha}{\rho}$ of $\mathfrak{sl}(2,\mathbb{C})$ with spin α, thus $\overset{\alpha}{\rho}$ admits a central character which we denote by λ_α. Given a framed knot K Let $J^\alpha(K)$ denote the framed Coloured Jones Function of it. Notice we "colour" the

Jones polynomial with the spin of the representation, rather than with the dimension of it. The last one is the usual convention.

Theorem 7 *We have:*

$$\frac{J^{\alpha}(K)}{2\alpha+1} = (\lambda_{\alpha} \circ \mathcal{Z}_t)(K), \forall \alpha \in \frac{1}{2}\mathbb{N}_0. \tag{3}$$

This is well known, though non trivial. A path for proving it relies upon the framework of quasi Hopf algebras and the notion of gauge transformations on them. This is described by Drinfeld in [D1] and [D2]. The context in which we need to apply it is the quantised universal enveloping algebras one. In this case some rigidity results ensure the above theorem is true. All this is described in detail in the same references. For a complete discussion, see [LM1] or[K].

1.4.4 Melvin-Morton Theorem and z-Coloured Jones Polynomial

Let K be a framed knot write

$$\frac{J^{\alpha}(K)}{2\alpha+1} = \sum_{n\in\mathbb{N}_0} J^{\alpha}_n(K) h^n.$$

It is a well known result that given a knot K then $J^{a}_n(K)$ is a polynomial in α with degree at most $2n$, cf [MM], [C]. This is known as Melvin-Morton theorem. The fact it is a polynomial is consequence of the fact that the centre of $U(\mathfrak{sl}(2,\mathbb{C}))$ is generated by the Casimir element of it, together with part 1 of Theorem 2. Notice the Casimir element act as $\alpha(\alpha+1)/2$ in the representation of spin α. The fact it is of degree $2n$ is clear from our discussion in 1.4.6. Alternatively we can prove it from part 2 of theorem 2 [2]. We will go back to this in 3.3. Therefore we can write:

$$\frac{J^{\alpha}(K)}{2\alpha+1} = \sum_{n\in\mathbb{N}_0}\sum_{k=0}^{2n} J_{n,k}(K)\alpha^k h^n, \tag{4}$$

where the constants $J_{n,k}(K)$ are uniquely determined, therefore defining a framed knot invariant. For any complex number z it thus makes sense to consider the z-coloured Jones polynomial:

$$\frac{J^{z}(K)}{2z+1} = \sum_{n\in\mathbb{N}_0} J^z_n(K)(z)h^n.$$

This yields thus a knot invariant whose value in a knot is a formal power series in two variables:

$$K \mapsto \sum_{m,n\in\mathbb{N}_0} J_{n,k}(K)z^k h^n.$$

with $J_{n,k}(K) = 0$ for $k > 2n$. A major part of this thesis aims to investigate whether or not this kind of series defines an analytic function in two variables. As we mentioned in the introduction, and are going to prove in 4.1, they can have a zero radius of convergence, thus this can only be made precise in a perturbation theory point of view, as we will see in 4.2. This relates to the question of whether it is possible to define *numerical* knot invariants out of the infinite dimensional representations of $SL(2,\mathbb{R})$, see theorem 26, and of the Lorentz Group after Theorem 16. It is known that if α is a half integer then

$$\frac{J^{\alpha}(K)}{2\alpha+1} = \sum_{m\in\mathbb{N}_0}\left(\sum_{k=0}^{2n} J_{n,k}(K)\alpha^k\right)h^n$$

[2]The part of the centre of $U(\mathfrak{sl}(2,\mathbb{C}))$ of degree smaller than $2n$ is generated by polynomials in the Casimir element of $\mathfrak{sl}(2,\mathbb{C})$ having degree n, see [VAR] for example

defines an analytic function in h. See the comments at the end of 1.3.2 and 3.3.

1.4.5 First example: The Unknot

For the unknot O the series $J^z(O)/(2z+1)$ has a non zero radius of convergence at any point $z \in \mathbb{C}$. The proof is not very difficult for we can have an explicit expression for it. Define, for each $z \in \mathbb{C}$ the meromorphic function:

$$F_z(h) = \frac{1}{2z+1} \frac{\sinh((2z+1)h/2)}{\sinh(h/2)}.$$

Thus for each $\alpha \in \frac{1}{2}\mathbb{N}_0$ we have

$$F_\alpha(h) = \frac{J^\alpha(O)}{2\alpha+1} = \sum_{n \in \mathbb{N}_0} J^\alpha_n(O) h^n.$$

Consider the expansion:

$$F_z(h) = \sum_{n \in \mathbb{N}_0} c(z)_n h^n.$$

It is not difficult to conclude that each $c(z)_n$ is a polynomial in z for fixed n. Moreover $c(\alpha)_n = J^\alpha_n(O), \forall \alpha \in \frac{1}{2}\mathbb{N}_0$. This implies

$$\frac{J^z(O)}{2z+1} = F_z(h),$$

as power series in h. In particular the power series for the unknot are convergent. This means it makes sense to speak about the quantum dimension of the representations of spin z, which are going to be defined later. To be more precise we made sense of the quantum dimension of them divided by their dimension as vector spaces.

But notice the dimension of a representation of spin z with $z \notin \frac{1}{2}\mathbb{N}_0$ is infinite. For some more explicit examples see 3.3.

1.4.6 A representation interpretation of the z-Coloured Jones Polynomial

We can give an interpretation of the z-coloured Jones polynomial in the framework of central characters. To this end, define the following elements of $\mathfrak{sl}(2, \mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, H = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the infinitesimal R-matrix which we are considering in $\mathfrak{sl}(2, \mathbb{C})$ expresses in the form:

$$t = -\frac{1}{4}\left(E \otimes F + F \otimes E + \frac{H \otimes H}{2}\right).$$

Notice that t is defined out of the inner product in $\mathfrak{sl}(2, \mathbb{C})$ which is minus the Cartan-Killing form. In particular, the Casimir element C of $\mathfrak{sl}(2, \mathbb{C})$ is equal to $-C_t$, where C_t is the quadratic central element associated with t. Recall subsection 1.3. The exact value of the Casimir element of $U(\mathfrak{sl}(2, \mathbb{C}))$ in this basis is:

$$C = \frac{1}{4}\left(EF + FE + \frac{H^2}{2}\right)$$

Given a half integer α, the representation space $\overset{\alpha}{V}$ of the representation of spin α has a basis of the form $\{v_0, ..., v_{2\alpha}\}$. The action of the elements E, F and H of $\mathfrak{sl}(2, \mathbb{C})$ in $\overset{\alpha}{V}$ is:

$$Hv_k = (k - \alpha)v_k,$$

$$Ev_k = (2\alpha - k)v_{k+1},$$

$$Fv_k = kv_{k-1}.$$

For a complex number $z \notin \frac{1}{2}\mathbb{N}_0$, it makes sense also to speak about the representation $\overset{z}{\rho}$ of spin z. Consider $\overset{z}{V}$ as being the infinite dimensional vector space which has the basis $\{v_{2z}, v_{2z-1}, v_{2z-2}, ...\}$. Then the representation $\overset{z}{\rho}$ of spin z is defined in the form:

$$Hv_k = (k - z)v_k; k = 2z, 2z - 1, ...,$$

$$Ev_k = (2z - k)v_{k+1}; k = 2z, 2z - 1, ...,$$

$$Fv_k = kv_{k-1}; k = 2z, 2z - 1, ...$$

The representations of spin $z \notin \frac{1}{2}\mathbb{N}_0$ have a central character λ_z, since it is easily proved that each intertwiner $\overset{z}{V} \to \overset{z}{V}$ must be a multiple of the identity. But see [VAR], 4.10.2. and 3.3, namely they are the unique cyclic highest weight representations with maximal weight z, this relative to the usual Borel decomposition of $\mathfrak{sl}(2, \mathbb{C})$. Consider, given $z \in \mathbb{C}$, the framed knot invariant $(\lambda_z \circ \mathcal{Z}_t)$. Where, if α is half integer, λ_α is the central character of the usual representation of spin α. See 1.3.3. Given a framed knot K it has the form:

$$(\lambda_z \circ \mathcal{Z}_t)(K) = \sum_{n \in \mathbb{N}_0} R_n^z(K)h^n,$$

where, by definition:

$$R_n^z(K) = (\lambda_z \circ \phi_t)(w_n) = \sum_{n \in \mathbb{N}_0} \lambda_z(\phi_t(w_n))h^n,$$

for

$$\mathcal{Z}(K) = \sum_{n \in \mathbb{N}_0} w_n, w_n \in \mathcal{A}_n, \forall n \in \mathbb{N}_0.$$

Also

$$\frac{J^\alpha(K)}{2z+1} = \sum_{n \in \mathbb{N}_0} R_n^\alpha(K) h^n, \forall \alpha \in \frac{1}{2}\mathbb{N}$$

Suppose w is a chord diagram with n chords. Let us have a look at the dependence of $\lambda_z(\phi_t(w))$ in z. It is not difficult to conclude that it is a polynomial in this variable of degree at most $2n$. This is a trivial consequence of the definition of the central element $\phi_t(w)$ as well as the kind of action of the terms appearing in the infinitesimal R-matrix t in $\overset{z}{\mathrm{V}}$. For more details see 6.2.1 and 3.3. In particular if K is a framed knot then $R_n^z(K)$ is a polynomial in z. Since by theorem 7 we also have $R_n^\alpha(K) = J_n^\alpha(K), \forall \alpha \in \frac{1}{2}\mathbb{N}_0$, we can conclude:

$$\frac{J^z(K)}{2z+1} = (\lambda_z \circ \mathcal{Z})(K),$$

which gives us an equivalent definition of the z-coloured Jones Polynomial.

The central characters of the representations of imaginary spin are actually the infinitesimal characters, cf [Kir] and 3.1, of the unitary representations of $SL(2,\mathbb{R})$ in the principal series, cf [L], with the same parameter. See 5.3 for the proof of this. Notice however that the derived representation of them in $\mathfrak{sl}(2,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C})$ is not any of the representation of imaginary spin just defined. We will consider this point of view later in 3.3.

2 The Lorentz Group and knot invariants

Let $\mathfrak{g}$ be a semisimple Lie Algebra. As proved by Drinfeld in [D1], there is a one to one correspondence between gauge equivalence classes of quasitriangular quantised universal enveloping algebras $\mathcal{H}$ of $\mathfrak{g}$ over $\mathbb{C}[[h]]$, cf [K], and infinitesimal R-matrices in $\mathfrak{g}$. Let us be more explicit about this. It is implicit in the definition of a quantised universal enveloping algebra $\mathcal{H}$ that there exists a $\mathbb{C}$-algebra morphism $f : \mathcal{H}/h\mathcal{H} \to U(\mathfrak{g})$. Having chosen such morphism, the canonical 2-tensor of $\mathcal{A}$ is defined as $t = f((R_{21}R - 1)/h)$. It is an infinitesimal R-matrix of $\mathcal{A}$. See [D1]. Here R denotes the universal R-matrix of $\mathcal{H}$. In the case of $U_h(\mathfrak{sl}(2,\mathbb{C}))$ this tensor t is the infinitesimal R-matrix in $\mathfrak{sl}(2,\mathbb{C})$ coming from the Cartan Killing form. The same happens for the Drinfeld-Jimbo quantisation of any semisimple Lie algebra. See [K]. Actually, if $\mathcal{H}$ quantises the pair $(\mathfrak{g}, r)$ where r is a classical r-matrix in $\mathfrak{g}$, see [CP], then t is the symmetrisation of r.

In general, each quantised universal enveloping algebra can be given a structure of a ribbon quasi Hopf algebra (as defined in [AC] for example), and therefore there is a knot invariant attached each finite dimensional representation of it, or what is the same, of $\mathfrak{g}$. These knot invariants take their values in the ring of formal power series over $\mathbb{C}$. If the representation used is finite dimensional and irreducible then it has a central character. In particular the framework of last section can be applied, using the infinitesimal R matrix t which is the canonical 2 tensor of $\mathcal{H}$. It is a deep result that with these choices the two approaches for knot invariants are the same, up to division by the dimension of the representation considered. This is a generalisation of theorem 37 to arbitrary semisimple Lie algebras. To be more precise we need also to change the sign of the infinitesimal R-matrix t, cf [K].

In the case in which we consider a q-deformation of the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, then no such classification of gauge equivalence classes of quantised universal enveloping algebras exists. But sometimes it is possible to make sense of the formula for t. This is because we have a q-parametrised family of *braided* Hopf algebras that tends to the universal enveloping algebra of $\mathfrak{g}$ as q goes to 1. Alternatively they can be seen as *particularisations* of the $\mathbb{C}[[h]]$-algebras $U_h(\mathfrak{g})$ to complex q.

As we mentioned in the introduction, despite the fact that the q-deformed Drinfeld-Jimbo quantised universal enveloping algebras $U_q(\mathfrak{g})$ of semisimple Lie algebras are not honest ribbon Hopf algebras, their category of finite dimensional representations is a ribbon category. That is they have formal R-matrices and ribbon elements, which make sense when acting in their finite dimensional representations. We will see this in chapter 7. The target space for the knot invariants in this context is the complex plane. These *numerical* knot invariants can be obtained, apart from rescaling, by summing the powers series which appear in context of h-adic Drinfeld-Jimbo algebras; in other words by summing the power series that come out of the approach making use the Kontsevich Integral and using the infinitesimal R-matrix which is the heuristic canonical 2-tensor t of $U_q(\mathfrak{g})$.

Let us pass now to the quantum Lorentz Group $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ as defined in [BR1] and [BR2]. It is a quantum group depending on a parameter $q \in (0,1)$. We will make a complete study of it in chapter 7. As said in the introduction, we wish to analyse the question of whether or not there exists a knot theory attached to the infinite dimensional representations of the Quantum Lorentz Group. The situation is more or less the same of the case of q-Drinfeld-Jimbo algebras. Namely we have a formal R-matrix which comes from its structure of a quantum double as well as a formal ribbon element. It is possible to describe how they act in the unitary representations of $\mathcal{D}$. The situation is simpler if the minimal spin of the representation is equal to zero, in which case the representation is said to be balanced. Representations of this kind are called simple in [NR]. In this context, the ribbon element acts as the identity and therefore the knot invariants obtained are unframed. Since we have a formal R-matrix and a formal ribbon element, given a knot diagram it is thus possible to make a formal Reshetikhin-Turaev evaluation of a knot invariant. This will yield an infinite sum for any of those. We will give a description of this construction in Chapter 7.

One natural thing to do would be analysing whether the "derivatives" of these sums in zero define or not Vassiliev invariants, or whether is possible to make sense of them, in the framework of Kontsevich Universal Invariant. It is not difficult to find an expression for an heuristic canonical 2-tensor of the quantum Lorentz group. Also the unitary representations of the Quantum Lorentz Group in the principal and complementary series have a classical counterpart. They are infinite dimensional representations of the Lie algebra of the Lorentz group which admit a central character, and therefore the framework of the last section can be used. This is the program we wish to consider now. We will prove that the knot invariants that we obtain are the ones defined out the infinite dimensional representations of the Quantum Lorentz Group[3] later in Chapter 7.

[3]Or more precisely the asymptotic expansion of these

2.1 The Lorentz Algebra

Consider the complex Lie group $SL(2,\mathbb{C})$. Its Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ is a complex Lie algebra of dimension 3. A basis of $\mathfrak{sl}(2,\mathbb{C})$ is $\{\sigma_X, \sigma_Y, \sigma_Z\}$ where

$$\sigma_Z = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \sigma_X = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \sigma_Y = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The commutation relations are:

$$[\sigma_X, \sigma_Y] = \sigma_Z, \quad [\sigma_Y, \sigma_Z] = \sigma_X, \quad [\sigma_Z, \sigma_X] = \sigma_Y.$$

We can also consider a different basis $\{H_+, H_-, H_3\}$, where

$$H_+ = i\sigma_X - s_Y, \quad H_- = i\sigma_X + i\sigma_Y, \quad H_3 = i\sigma_Z.$$

The new commutation relations being:

$$[H_+, H_3] = -H_3, \quad [H_-, H_3] = H_-, \quad [H_+, H_-] = 2H_3.$$

Restricting the ground field with which we are working to $\mathbb{R}$, we obtain the 6 dimensional real Lie algebra $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, the realification of $\mathfrak{sl}(2,\mathbb{C})$.

Definition 8 *The Lorentz Lie algebra L is defined as being the complex Lie algebra which is the complexification of $\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$. In other words $L = \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$. It is therefore a complex Lie algebra of dimension 6. The Lorentz algebra is the complex algebra $U(L)$ which is the universal enveloping algebra of the complex Lie algebra L.*

The set $\{\sigma_X, B_X = -i\sigma_X, \sigma_Y, B_Y = -i\sigma_Y, \sigma_Z, B_Z = -i\sigma_Z\}$ is a real basis of

$\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}$, and thus a complex basis of L. The commutation relations are:

$$\begin{array}{lll}
[\sigma_X,\sigma_Y]=\sigma_Z & [\sigma_Y,\sigma_Z]=\sigma_X & [\sigma_Z,\sigma_X]=\sigma_Y \\
[\sigma_Z,B_X]=B_Y & [\sigma_Y,B_X]=-B_Z & [\sigma_X,B_X]=0 \\
[\sigma_Z,B_Y]=-B_X & [\sigma_Y,B_Y]=0 & [\sigma_X,B_Y]=B_Z \\
[\sigma_Z,B_Z]=0 & [\sigma_Y,B_Z]=B_X & [\sigma_X,B_Z]=-B_Y \\
[B_X,B_Y]=-\sigma_Z & [B_Y,B_Z]=-\sigma_X & [B_Z,B_X]=-\sigma_Y.
\end{array}$$

We can also consider the basis $\{H_+,H_-,H_3,F_+,F_-,F_3\}$ of L, where :

$$H_+=i\sigma_X-\sigma_Y,\quad H_-=i\sigma_X+\sigma_Y,\quad H_3=i\sigma_Z,$$

$$F_+=iB_X-B_Y,\quad F_-=iB_X+B_Y,\quad F_3=iB_Z.$$

The new commutation relations being:

$$[H_+,H_3]=-H_+,[H_-,H_3]=-H_-,[H_+,H_-]=2H_3,$$

$$[F_+,H_+]=[H_-,F_-]=[H_3,F_3]=0,$$

$$[H_+,F_3]=-F_+,[H_-,F_3]=F_-,$$

$$[H_+,F_-]=-[H_-,F_+]=2F_3,$$

$$[F_3,H_3]=-F_+,[F_-,H_3]=F_-,$$

$$[F_+,F_3]=H_+,[F_-,F_3]=H_-,[F_+,F_-]=-2H_3.$$

The following simple theorem will be one of the most important in our discussion.

Theorem 9 *There exists one (only) isomorphism of complex Lie algebras* $\tau : \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \to L \cong \mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ *such that:*

$$\sigma_X \oplus 0 \mapsto \frac{\sigma_X - i\sigma_X \otimes i}{2} = \frac{\sigma_X + iB_X}{2},$$

$$0 \oplus \sigma_X \mapsto \frac{\sigma_X + i\sigma_X \otimes i}{2} = \frac{\sigma_X - iB_X}{2},$$

$$\sigma_Y \oplus 0 \mapsto \frac{\sigma_Y - i\sigma_Y \otimes i}{2} = \frac{\sigma_Y + iB_Y}{2},$$

$$0 \oplus \sigma_Y \mapsto \frac{\sigma_Y + i\sigma_Y \otimes i}{2} = \frac{\sigma_Y - iB_Y}{2},$$

$$\sigma_Z \oplus 0 \mapsto \frac{\sigma_Z - i\sigma_Z \otimes i}{2} = \frac{\sigma_Z + iB_Z}{2},$$

$$0 \oplus \sigma_Z \mapsto \frac{\sigma_Z + i\sigma_Z \otimes i}{2} = \frac{\sigma_Z - iB_Z}{2}.$$

And thus we have also a Hopf algebra isomorphism

$$\tau : U(\mathfrak{sl}(2,\mathbb{C})) \otimes U(\mathfrak{sl}(2,\mathbb{C})) \to U(L).$$

Proof. Easy calculations ■

Given $X \in \mathfrak{sl}(2,\mathbb{C})$, define $X^l = \tau(X \oplus 0), X^r = \tau(0 \oplus X)$. And analogously for $X \in U(\mathfrak{sl}(2,\mathbb{C}))$. We have:

$$H_+^l = \frac{H_+ + iF_+}{2}, \quad H_-^l = \frac{H_- + iF_-}{2}, \quad H_+^l = \frac{H_3 + iF_3}{2},$$

$$H_+^r = \frac{H_+ - iF_+}{2}, \quad H_-^r = \frac{H_- - iF_-}{2}, \quad H_+^r = \frac{H_3 - iF_3}{2}.$$

Consider also $C^l = \tau(C \otimes 1)$ and $C^r = \tau(1 \otimes C)$, where C is the Casimir element of $\mathfrak{sl}(2,\mathbb{C})$ defined in 1.4.6. The elements C^l and C^r are called Left and Right Casimirs and their explicit expression is:

$$
\begin{aligned}
4C^l = \frac{H_3^2 - F_3^2}{2} + i\frac{H_3F_3}{2} + i\frac{F_3H_3}{2} \\
+ \frac{H_+H_-}{4} + i\frac{H_+F_-}{4} + i\frac{F_+H_-}{4} - \frac{F_+F_-}{4} \\
+ \frac{H_-H_+}{4} + i\frac{H_-F_+}{4} + i\frac{F_-H+}{4} - \frac{F_-F_+}{4},
\end{aligned}
$$

and

$$
\begin{aligned}
4C^r = \frac{H_3^2 - F_3^2}{2} - i\frac{H_3F_3}{2} - i\frac{F_3H_3}{2} \\
+ \frac{H_+H_-}{4} - i\frac{H_+F_-}{4} - i\frac{F_+H_-}{4} - \frac{F_+F_-}{4} \\
+ \frac{H_-H_+}{4} - i\frac{H_-F_+}{4} - i\frac{F_-H+}{4} - \frac{F_-F_+}{4}.
\end{aligned}
$$

We can also consider the left and right image under $\tau \otimes \tau$ of the infinitesimal R-matrix of $U(\mathfrak{sl}(2,\mathbb{C}))$. We take now $t \in \mathfrak{sl}(2,\mathbb{C}) \otimes \mathfrak{sl}(2,\mathbb{C})$ as being the infinitesimal R-matrix coming from the Cartan-Killing form. That is minus the one considered in 37. These left and right infinitesimal R-matrices are:

$$
\begin{aligned}
4t^l = \frac{H_3 \otimes H_3}{2} - \frac{F_3 \otimes F_3}{2} + i\frac{H_3 \otimes F_3}{2} + i\frac{F_3 \otimes H_3}{2} \\
+ \frac{H_+ \otimes H_-}{4} + i\frac{H_+ \otimes F_-}{4} + i\frac{F_+ \otimes H_-}{4} - \frac{F_+ \otimes F_-}{4} \\
+ \frac{H_- \otimes H_+}{4} + i\frac{H_- \otimes F_+}{4} + i\frac{F_- \otimes H+}{4} - \frac{F_- \otimes F_+}{4},
\end{aligned}
$$

and

$$4t^r = \frac{H_3 \otimes H_3}{2} - \frac{F_3 \otimes F_3}{2} - i\frac{H_3 \otimes F_3}{2} - i\frac{F_3 \otimes H_3}{2}$$
$$+ \frac{H_+ \otimes H_-}{4} - i\frac{H_+ \otimes F_-}{4} - i\frac{F_+ \otimes H_-}{4} - \frac{F_+ \otimes F_-}{4}$$
$$+ \frac{H_- \otimes H_+}{4} - i\frac{H_- \otimes F_+}{4} - i\frac{F_- \otimes H+}{4} - \frac{F_- \otimes F_+}{4}.$$

Any linear combination $at^l + bt^r$ of the left and right infinitesimal R-matrices is an infinitesimal R-matrix for L. In fact all them are of this form as proved in 1.3.1. We wish to consider the combination $t_L = t^l - t^r$. That is

$$t_L = i\frac{1}{4}H_3 \otimes F_3 + \frac{1}{4}iF_3 \otimes H_3 + \frac{i}{8}H_- \otimes F_+ + \frac{i}{8}F_- \otimes H_+ + \frac{i}{8}H_+ \otimes F_- + \frac{i}{8}F_+ \otimes H_-.$$

Notice another expression of it:

$$t_L = \frac{i}{8}\left(B_X \otimes \sigma_X + \sigma_X \otimes B_X + B_Y \otimes \sigma_Y + \sigma_Y \otimes B_Y + B_Z \otimes \sigma_Z + \sigma_Z \otimes B_Z\right).$$

The quadratic central element of $U(L)$ associated with t_L is:

$$C_L = C_{t_L} = i\frac{H_3F_3}{4} + i\frac{F_3H_3}{4} + i\frac{H_+F_-}{8} + i\frac{F_+H_-}{8} + i\frac{H_-F_+}{8} + i\frac{F_-H_+}{8}$$

The reason why we consider this particular combination of the left and right infinitesimal R-matrices is because it corresponds to the heuristic canonical two tensor of the Quantum Lorentz group $U_q(\mathfrak{sl}(2,\mathbb{C})_\mathbb{R})$ considered in [BR2]. Notice it is the symmetrisation of the classical r-matrix of $\mathfrak{sl}(2,\mathbb{C})_\mathbb{R}$, see [BNR] page 19. The knot theory obtained from t_L and the unitary representations of the Lorentz Group ought to be related to any knot invariants that can be defined from the unitary representations of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ for $q \in (0,1)$. We shall see in the last chapter (Theorem 136) that this is the case, at least for balanced representations.

The general case in which we admit other infinitesimal R-matrices is not more difficult, however the relations with the knot theory coming from the Quantum Lorentz Group is not as transparent. This is because we are changing the quantum group with which we are working. See the introduction to this chapter. Another interesting combination of the Left and Right infinitesimal R-matrices is the following:

$$\hat{t}_L = t^l + t^r = \frac{1}{8}(\sigma_X \otimes \sigma_X - B_X \otimes B_X + \sigma_Y \otimes \sigma_Y - B_Y \otimes B_Y + \sigma_Z \otimes \sigma_Z - B_Z \otimes B_Z). \quad (5)$$

It corresponds to working with $U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$, with q in the unit circle.

Both t_L and $\hat{t}_L$ are identified with the obvious invariant non-degenerate bilinear forms in the Lie algebra of the Lorentz group [4]. In fact, it is possible to prove that the Chern-Simons functional

$$CS(A) = \exp\left(i\frac{1}{4\pi}\int_M \mathrm{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)\right) \quad (6)$$

is gauge invariant if and only if tr is defined out of the non-degenerate, invariant, bilinear form in L associated with $n\hat{t}_L + st_L$, with $n \in \mathbb{Z}$ and $s \in \mathbb{C}$, see [W]. Here A denotes an L-valued 1-form in a 3 manifold M.

[4] The first is the imaginary part of the Cartan Killing form in $\mathfrak{sl}(2,\mathbb{C})$ whereas the second is the real part of it

2.1.1 The irreducible Balanced Representations of the Lorentz Group

Let us be given a complex number $p = |p|e^{i\theta}, 0 \leq \theta < 2\pi$ different from zero. We *define* once for all[5] the square root $\sqrt{p}$ of p as being $\sqrt{|p|e^{i\theta}} = \sqrt{|p|}e^{i\frac{\theta}{2}}$. For $m \in \mathbb{Z}$ define the set $W_m = \{p \in \mathbb{C} : |p| \notin \mathbb{N}_{|m|+1}\}$, where, in general, $\mathbb{N}_m = \{m, m+1, ...\}$, for any $m \in \mathbb{N}$. Consider the set $\mathcal{P} = \{(m, p) : m \in \mathbb{Z}, p \in W_m\}$. Define, for any $\alpha \in \mathbb{N}$ and $(p, m) \in \mathcal{P}$:

$$C_\alpha(p,m) = \frac{i}{\alpha}\sqrt{\frac{(\alpha^2 - p^2)(\alpha^2 - m^2)}{4\alpha^2 - 1}},$$

$$B_\alpha(p,m) = \frac{ipm}{\alpha(\alpha+1)}.$$

Thus $C_\alpha(p, m) \neq 0, \alpha = |m| + 1, |m| + 2..., \forall p \in W_m$

Consider the complex vector space

$$V(m) = \bigoplus_{\alpha \in \mathbb{N}_{|m|}} \overset{\alpha}{V},$$

where $\overset{\alpha}{V}$ denotes the representation space of the representation of $\mathfrak{sl}(2, \mathbb{C})$ of spin α. The set $\{\overset{\alpha}{v}_i, i = -\alpha, -\alpha + 1, ..., \alpha; \alpha \in \mathbb{N}_{|m|}\}$ is a basis of $V(m)$. Consider the inner product in $V(m)$ that has the basis above as an orthonormal basis. Define also $\bar{V}(m)$ as being the Hilbert space which is the completion of $V(m)$.

Given $(p, m) \in \mathcal{P}$, consider the following linear operators acting on $V(m)$:

$$H_3\overset{\alpha}{v}_k = k\overset{\alpha}{v}_k, k = -\alpha, -\alpha + 1, ..., \alpha, \alpha \in \mathbb{N}_{|m|},$$

$$H_-\overset{\alpha}{v}_k = \sqrt{(\alpha + k)(\alpha - k + 1)}\overset{\alpha}{v}_{k-1}, k = -\alpha, -\alpha + 1, ..., \alpha$$

[5]We will see in a second that this choice is totally irrelevant

$$H_+\overset{\alpha}{v}_k = \sqrt{(\alpha+k+1)(\alpha-k)}\overset{\alpha}{v}_{k+1}, k = -\alpha, -\alpha+1, ..., \alpha,$$

$$\begin{aligned} F_+\overset{\alpha}{v}_k = C_\alpha(m,p)\sqrt{(\alpha-k)(\alpha-k-1)} \quad & \overset{\alpha-1}{v}_{k+1} \\ & - B_\alpha(m,p)\sqrt{(\alpha+k+1)(\alpha-k)}\overset{\alpha}{v}_{k+1} \\ & + C_{\alpha+1}(m,p)\sqrt{(\alpha+k+1)(\alpha+k+2)} \quad \overset{\alpha+1}{v}_{k+1}, \\ & k = -\alpha, -\alpha+1, ..., \alpha, \alpha \in \mathbb{N}_{|m|}, \end{aligned}$$

$$\begin{aligned} F_-\overset{\alpha}{v}_k = -C_\alpha(m,p)\sqrt{(\alpha+k)(\alpha+k-1)} \quad & \overset{\alpha-1}{v}_{k-1} \\ & - B_\alpha(m,p)\sqrt{(\alpha-k+1)(\alpha+k)}\overset{\alpha}{v}_{k+1} \\ & - C_{\alpha+1}(m,p)\sqrt{(\alpha+1)^2-k^2} \quad \overset{\alpha+1}{v}_{k}, \\ & k = -\alpha, -\alpha+1, ..., \alpha, \alpha \in \mathbb{N}_{|m|}, \end{aligned}$$

$$\begin{aligned} F_3\overset{\alpha}{v}_k = C_\alpha(m,p)\sqrt{\alpha^2-k^2} \quad & \overset{\alpha-1}{v}_{k} - B_\alpha(m,p)k\overset{\alpha}{v}_k \\ & - C_{\alpha+1}(m,p)\sqrt{(\alpha+1)^2-k^2} \quad \overset{\alpha+1}{v}_{k}, \\ & k = -\alpha, -\alpha+1, ..., \alpha, \alpha \in \mathbb{N}_{|m|}. \end{aligned}$$

Obviously we are considering $\overset{\alpha}{v}_k = 0$ if $k > \alpha$ or $k < -\alpha$. We have the following theorem, whose proof can be found in [GMS]

Theorem 10 *If $(p,m) \in \mathcal{P}$, the operators $H_-, H_+, H_3, F_-, F_+, F_3$ define an infinite dimensional representation of the Lorentz algebra.*

Notice that the representations (m,p) and $(-m,-p)$ are equivalent. This has a trivial proof.

In fact, the operators displayed before define a representation of the Lorentz algebra if and only if:

$$\begin{cases} (B_\alpha(\alpha+1)-(\alpha-1)B_{\alpha-1})\, C_\alpha = 0, \\ (B_{\alpha+1}(\alpha+2)-\alpha B_\alpha)\, C_{\alpha+1} = 0, \\ (2\alpha-1)C_\alpha^2-(2\alpha+3)C_{\alpha+1}^2-B_\alpha^2 = 1. \end{cases} \tag{7}$$

The quantum case will be very similar. We have put $B_a(m,p) = B_\alpha$ and $C_\alpha(m,p) = C_\alpha$ to simplify. See [GMS]. In particular the square root we choose in the definition of $C_\alpha(m,p)$ is not important if we want to define a representation. In addition, making the transformation $\overset{\alpha}{v}_i \mapsto (-1)^{c_\alpha}\overset{\alpha}{v}_i$ where $c_\alpha \in \{0,1\}$ tells us the choice of the square root does not affect the representation apart from isomorphism, therefore being totally irrelevant.

Denote the representations above by $\{\rho(m,p) : (m,p) \in \mathcal{P}\}$. One can prove with no difficulty that they have a central character, for any intertwiner $V(m) \to V(m)$ needs to send each space $\overset{\alpha}{V}$ to itself and act on it has a multiple of the identity. Considering the action of F_+, for example, we conclude that the multiples are the same in each space $\overset{\alpha}{V}$. See [BR2], proof of irreducibility in theorem 3. Therefore

Theorem 11 *For any $(m,p) \in \mathcal{P}$ the representation $\rho(m,p)$ of L admits a central character $\lambda_{m,p}$.*

For any $(m,p) \in \mathcal{P}$ the representation $\rho(m,p)$ of L can always be integrated to a representation $R(m,p)$ of the Lorentz group in the completion $\bar{V}(m)$ of $V(m)$, or to be more precise of its connected component of the identity, see [GMS] $II.4.7$. The resulting representation is unitary if and only if p is purely imaginary, for any $m \in \mathbb{N}_0$, in which case the representation is said to belong to the principal series, or if $m = 0$ and $p \in [0,1)$ in which case the representation is said to belong to the complementary series. In other words, for these last cases, there exist a representation $R(m,p)$ of the Lorentz group in $\bar{V}(m)$ such that $V(m)$ is contained in the space $V_\infty(m,p)$ of smooth vectors of $R(m,p)$, and, moreover, such that the derived representation $R^\infty(m,p)$ of $U(L)$ in $V_\infty(m,p)$ when restricted to $V(m)$ is $\rho(m,p)$. see [Kir] or 3.1 for nomenclature and notation. See 3.3.3 for a description of the representations in the principal series described as representations of $SL(2,\mathbb{C})$, the universal covering of the unit component of the Lorentz group. Since the vector space $V(m)$ is contained in the space of smooth vectors $R(m,p)$, it is trivial to conclude that $\lambda_{m,p}$ is the infinitesimal character of $R(m,p)$. This unifies the approach here with the approach in 3.3.3.

The parameter m is called the minimal spin of the representation. A representation is called balanced[6] if the minimal spin of it is 0. Balanced representations depend therefore on a parameter $p \in W_0$. Denote them by $\{\rho_p, p \in W_0\}$. Two balanced representations ρ_p and ρ_q of L are equivalent if and only if $p = q$ or $p = -q$. These representations were used in [BC] for the construction of a spin foam model for Quantum Gravity. The extension of that work for their quantised counterpart was dealt with in [NR].

Observation 12 *Since the representations $\{\rho(m,p) : (m,p) \in \mathcal{P}\}$ have a central character, the left and right Casimirs defined in 2.1 act on $V(m)$ as multiples of the identity. This multiples are, as a function of m and p the following: $\frac{p^2+2mp+m^2-1}{8}$ for C^l and $\frac{p^2-2mp+m^2-1}{8}$ for C^r.*

Therefore:

[6] This nomenclature is due to John Barrett and Louis Crane. See [BC]

Proposition 13 *If the infinitesimal R-matrix on $U(L)$ is the tensor t_L defined in 2.1 then the central characters of the balanced representations $\{\lambda_p, p \in W_0\}$ are t_L-unframed. Recall the nomenclature introduced before theorem 4.*

This can obviously be proved without using the explicit expression of the action of the Casimir elements.

Notice also that we can consider the minimal spin of the representations considered to be also to be an half integer, making the obvious change in the form of the representation. These kind of representations cannot be integrated to representations of the Lorentz group, even though they define representations of $SL(2, \mathbb{C})$. They are called two-valued representations of the Lorentz group in [GMS] .

2.2 The Lorentz Knot Invariant

Consider again the infinitesimal R-matrix $t_L = t^l - t^r$ of the Lorentz Lie algebra. We consider for each $(m, p) \in \mathcal{P}$ the representation $\rho(m, p)$ of L. It has a central character $\lambda_{m,p}$. We propose to consider the framed knot invariants $\{X(m, p) : (m, p) \in \mathcal{P}\}$, such that for any knot:

$$K \mapsto X(m, p, K) = (\lambda_{m,p} \circ \mathcal{Z}_{t_l})(K) = (\lambda_{m,p} \circ \phi_{t_L} \circ \mathcal{Z})(K).$$

Recall the notation of 1.3. Notice $X(m, p) = X(-m, -p)$ for the representations $\rho_{m,p}$ and $\rho_{-m,-p}$ are equivalent.

The value of $X(m, p)$ in a framed knot K is therefore a formal power series with coefficients in $\mathbb{C}$. It is a difficult task to analyse the analytic properties of such power series. We expect they will be perturbation series for some numerical knot invariants that can be defined. This is the main subject of chapter 4.

As we have seen, if $m = 0$, that is in the case of balanced representations, the central character λ_p is t_L-unframed. This is also the case for $p = 0$. Notice we have an explicit expression for the action of the left and right Casimir elements of L. Therefore

Theorem 14 *The knot invariant $X(m,p)$ with $(m,p) \in \mathcal{P}$ is unframed if and only if $m = 0$ or $p = 0$.*

Obviously, for different combinations of the left and right infinitesimal R-matrices, the representations which have unframed central characters with respect to it are different. This gives us a way to define an unframed knot invariant out of any $(m,p) \in \mathcal{P}$. But notice this can be done without changing the infinitesimal R-matrix t_L of L, since we know how the invariants behave with respect to framing, cf theorem 1.

2.2.1 Finite Dimensional Representations

Let us now analyse the knot invariants that come out of the finite dimensional representations of the Lorentz group. We are mainly interested in the representations which are irreducible.

Since we have the isomorphism $U(L) \cong U(\mathfrak{sl}(2,\mathbb{C})) \otimes U(\mathfrak{sl}(2,\mathbb{C}))$, the finite dimensional irreducible representation of $U(L)$, or what is the same of L, are classified by a pair (α, β) of half integers. That is each finite dimensional irreducible representation of L is of the form $\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}$ as a representation of $U(L) \cong U(\mathfrak{sl}(2,\mathbb{C})) \otimes U(\mathfrak{sl}(2,\mathbb{C}))$. There is an alternative way to construct these finite dimensional representations that shows their close relation with the infinite dimensional representations, [GMS]. Let us explain how the process goes. It is very similar to the $\mathfrak{sl}(2,\mathbb{C})$ case.

Consider $m = \alpha - \beta$ and $p = \alpha + \beta + 1$. Notice that now $C_\alpha(m,p) \neq 0$ if $\alpha \in |m|, |m|+1, ..., p$, and $C_p(m,p) = 0$. The underlying vector space for the representation with spins (α, β) is $V(m,p) = \overset{|m|}{V} \otimes \overset{|m|+1}{V} \otimes ... \otimes \overset{p-1}{V}$, and the form of it is given exactly by the same formulae of the infinite dimensional representations. The equivalence of the representations is a trivial consequence of the Clebsch-Gordan formula and Observation 12 (which still holds) and the know action of the Casimir element of $\mathfrak{sl}(2,\mathbb{C})$ (see page 36); since these representations are irreducible. This construction gives us a finite dimensional representation $\rho(m,p)$ for each pair (m,p) with $m, p \in \mathbb{Z}/2$ and $p - |m| \in \mathbb{N}_1$. It makes also sense for $|p| - |m| \in \mathbb{Z}$, making the appropriate changes. As before we have the equivalence $\rho(m,p) \cong \rho(-m,-p)$.

Since we completed the sets W_m defined at the beginning of 2.1.1, we have a representation $\rho(m,p)$ of the Lorentz algebra for each pair (m,p) with $m \in \mathbb{Z}/2$ and $p \in \mathbb{C}$. All of them have a central character $\lambda_{m,p}$, since the new representations considered are finite dimensional and irreducible. The finite dimensional representations give us framed knot invariant $X_{\text{fin}}(m,p)$ for each pair $m, p \in \mathbb{Z}/2$ with $|p| - |m| \in \mathbb{N}_1$. This invariant is independent of the framing if and only if $m = 0$, that is if $\alpha = \beta$.

Consider now the algebra morphisms $\lambda_{m,p} \circ \phi_{t_L} : \mathcal{A} \to \mathbb{C}$, where $m \in \mathbb{Z}$ and $p \in \mathbb{C}$. The argument is now similar to the one in 1.4.6. If we look at the expression of the representations $\rho_{m,p}$, it is easy to conclude that given any chord diagram w with n chords, the evaluation of $\lambda_{m,p} \circ \phi_{t_L}(w)$ is for a fixed m a polynomial in p of degree at most $2n$. Notice that any factor of the form $C_\alpha(m,p)$ appears in the expression for $\lambda_{m,p} \circ \phi_{t_L}(w)$ an even number of times. For those who want more details, all this can be proved from Observation 12 and the fact the centre of $U(L)$ is the polynomial algebra in the left and right casimirs C^l and C^r. For the case of balanced representations, that is $m = 0$, we can also prove that it is a polynomial in p^2. Also the value of the polynomials in $p = 1$ is zero if $n > 0$ for the pair with $m = 0$ and $p = 1$ yields the trivial one dimensional representation of L.

We have proved:

Theorem 15 *Consider the framed knot invariants* $\{X(m,p), m \in \mathbb{N}_0, p \in \mathbb{C}\}$. *If we fix* $m \in \mathbb{N}_0$ *then the term of order* n *in the expansion of* $X(m,p,K)$, *where* K *is any framed knot, as a power series is polynomial of degree at most* $2n$ *in* p . *If* $m = 0$ *then only the even terms of it are non zero. Moreover the polynomials attain zero at* $p = 1$ *for* $n > 0$.

Therefore, if we know the value of $X(m,p,K)$ for the finite dimensional representations, that is if $|p| - |m| \in \mathbb{N}$ we can determine it for any value of the parameter p. This is similar to the $\mathfrak{sl}(2,\mathbb{C})$ case.

2.2.2 Relation with the Coloured Jones Polynomial

The relation between the Lorentz Knot invariants that come out from finite dimensional and infinite dimensional representations remarked after theorem 23 gives us a way to relate the coloured Jones Polynomial with the Lorentz group invariants. In fact:

Theorem 16 *Let* K *be some oriented framed knot,* K^* *its mirror image. Then for any* $z, w \in \mathbb{C}$ *with* $z - w \in \mathbb{Z}$ *we have:*

$$\frac{J^z(K^*)}{2z+1} \times \frac{J^w(K)}{2w+1} = X(z-w, z+w+1, K),$$

as formal power series over $\mathbb{C}$.

Proof. For any $m \in \mathbb{Z}/2$ and $x \in \mathbb{C}$, let $z(x,m) = m + x$ and $w(x,m) = -m + x$. Thus each pair $(z, w) \in \mathbb{C}^2$ with $z - w \in \mathbb{Z}$ is of the form $(z(x,m), w(x,m))$ for some m and x. Fix $m \in \mathbb{Z}/2$. We want to prove:

$$\frac{J^{(m+x)}(K^*)}{2m+2x+1} \times \frac{J^{(-m+x)}(K)}{-2m+2x+1} = X(m, 2x+1, K), \forall x \in \mathbb{C}.$$

Each term of the formal power series at both sides of the equality is a polynomial in x, thus we only need to prove that the equality is true if both $x - m$ and $x + m$ are half integers. That is if $x - m, x + m \in \frac{1}{2}\mathbb{N}_0$.

Let t be the infinitesimal R matrix in $\mathfrak{sl}(2, \mathbb{C})$ coming out of the Cartan-Killing form. Notice it is minus the one considered in 1.4.3. Let α be a half integer. Recall that for a framed knot K we have:

$$\frac{J^\alpha(K)}{2\alpha+1} = (\lambda_\alpha \circ \mathcal{Z}_{-t})(K).$$

Therefore by part 3 of Theorem 1:

$$\frac{J^\alpha(K^*)}{2\alpha+1} = (\lambda_\alpha \circ \mathcal{Z}_t)(K),$$

since $\phi_t(w) = (-1)^n \phi_{-t}(w)$ if w is a chord diagram with n chords.

Let K be a framed knot and x be such that $\alpha = x - m$ and $\beta = x + m$ are half integers. We have by theorem 5:

$$\frac{J^\alpha(K^*)}{2\alpha+1} \times \frac{J^\beta(K)}{2\beta+1} = (\lambda_\alpha \circ \mathcal{Z}_t)(K) \times (\lambda_\beta \circ \mathcal{Z}_{-t})(K)$$
$$= (\lambda_\alpha \circ \mathcal{Z}_{t^l})(K) \times (\lambda_\beta \circ \mathcal{Z}_{-t^r})(K)$$
$$= ((\lambda_\alpha \otimes \lambda_\beta) \circ \mathcal{Z}_{t_L})\,(K).$$

Recall $t_L = t^l - t^r$.

Now, $\lambda_\alpha \otimes \lambda_\beta$ is the central character of the representation $\rho_\alpha \otimes \rho_\beta$ of $U(L) \cong U(\mathfrak{sl}(2,\mathbb{C})) \otimes U(\mathfrak{sl}(2,\mathbb{C}))$. As we have seen before, this representation is equivalent to $\rho(\alpha-\beta, \alpha+\beta+1) = \rho(m, 2x+1)$. Thus their central characters are the same. This proves

$$((\lambda_\alpha \otimes \lambda_\beta) \circ \mathcal{Z}_{t_L})\,(K) = (\lambda_{m,2x+1} \circ \mathcal{Z}_{t_L})(K),$$

if both $x-m$ and $x+m$ are half integers, and the proof is finished. ■

We have the following simple consequences.

Corollary 17 *Given a framed knot K, then the term of order n in the power series of $X(m,z,K)$ is a polynomial in m and z*

Corollary 18 *If O is the unknot, then $X(m,p,O)$ is a convergent power series.*

Corollary 19 *For balanced representations, that is if $m=0$, the invariant $X(0,p)$ does not distinguish a knot from its mirror image.*

Corollary 20 *The framed knot invariants $X(m,p)$ are unoriented.*

Obviously Theorem 14 also follows from Theorem 16.

3 General non compact group knot invariants

We now give a general framework for defining knot invariants from unitary representations of Lie Groups, not assumed to be finite dimensional. As we will see the setting is going to be algebraic at the end.

3.1 Unitary representations and infinitesimal characters

Let G be a Lie group, always assumed to be real, and $\mathfrak{g}$ be its Lie algebra. Let also $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ denote the complexification of $\mathfrak{g}$. Consider a unitary representation R of G in the complex Hilbert space V. Notice V is not assumed to be finite dimensional. We recall that unitarity means that the operator $R(g) : V \to V$ is a unitary for any $g \in G$. Also we suppose a continuity condition, namely that for any $v \in V$ the map $g \in G \mapsto R(g)v \in V$ is continuous.

The main reference for what follows is [Kir]. Let V_∞ denote the vector space of smooth vectors of R. That is

$$V_\infty = \{v \in V : g \in G \mapsto R(g)(v) \in C^\infty(G, V)\}. \tag{8}$$

It is well known that V_∞ is dense in V. Differentiating R at the identity of G defines a map $R^\infty : \mathfrak{g} \otimes V_\infty \to V$. It is possible to show that V_∞ is invariant under $\mathfrak{g}$ and that R^∞ is a honest representation of $\mathfrak{g}$. It extends therefore to a representation, which we also call R^∞, of $U(\mathfrak{g}_{\mathbb{C}})$ in V_∞. Recall that V is a complex vector space.

Suppose R is a irreducible representation. In our context this means that V has no closed invariant subspaces under the action of G. The following (non trivial) result can be found in [Kir]:

Theorem 21 *If V is irreducible then any element of $\mathcal{C}(U(\mathfrak{g}_{\mathbb{C}}))$ acts on V_∞ under R^∞ as a multiple of the identity operator.*

Recall $\mathcal{C}(U(\mathfrak{g}_{\mathbb{C}}))$ denotes the centre of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

Obviously if V is a finite dimensional complex vector space then the theorem just stated is a consequence of Schur's lemma and the unitarity condition is not needed. Also it is in general possible to show directly that the above property is true for a large class of infinite dimensional representations, not necessarily unitary. Amongst them are the representations of $SL(2,\mathbb{R})$ and $SL(2,\mathbb{C})$ which we are going to consider. However the last theorem tells us that our construction is general.

Another way to state Theorem 21 is to say that R has an infinitesimal character. In other words there exists a (unique) central character χ_R^∞ of $U(\mathfrak{g}_{\mathbb{C}})$, that is a morphism of complex algebras $\mathcal{C}(U(\mathfrak{g}_{\mathbb{C}})) \to \mathbb{C}$, with the property:

$$R^\infty(a)(v) = \chi_R^\infty(a)v, \forall v \in V^\infty, \forall a \in \mathcal{C}(U(\mathfrak{g}_{\mathbb{C}})). \tag{9}$$

Therefore we have the following obvious conclusion (Cf 1.3.3):

Theorem 22 *Let G be a real Lie group and R an irreducible unitary representation of G in some complex Hilbert space V. Let also t be an infinitesimal R-matrix in the complexification $\mathfrak{g}_{\mathbb{C}}$ of the Lie algebra $\mathfrak{g}$ of G. There exists a knot invariant with values in $\mathbb{C}[[h]]$:*

$$I(G,t,R) = \chi_R^\infty \circ \phi_t \circ \mathcal{Z}. \tag{10}$$

It has the form

$$K \mapsto \sum_{n=0}^{+\infty} \chi_R^\infty(\phi_t \circ \mathcal{Z}_n(K))h^n. \tag{11}$$

As an example, consider $G = SU(2)$, thus $\mathfrak{g} = \mathfrak{su}(2)$ and $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. Take t to be the infinitesimal R-matrix coming from minus the Cartan-Killing form in $\mathfrak{sl}(2, \mathbb{C})$, that is $< X, Y >= -\mathrm{tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$. The tensor t has the form:

$$t = -\frac{1}{4}(\sigma_X \otimes \sigma_X + \sigma_Y \otimes \sigma_Y + \sigma_Z \otimes \sigma_Z), \tag{12}$$

where

$$\sigma_X = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \sigma_Y = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \sigma_Z = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{13}$$

The following is a restatement of Theorem 7:

Theorem 23 *Let J^α denote the framed coloured Jones polynomial associated with the representation R^α of $SU(2)$ with spin $\alpha \in \{0, 1/2, 1, 3/2...\}$. We take the normalisation of the coloured Jones polynomial that evaluates at the unknot to the quantum dimension of the $U_h(\mathfrak{sl}(2, \mathbb{C}))$ spin α representation. Given any framed knot K we have:*

$$\frac{J^\alpha}{2\alpha + 1}(K) = I(SU(2), t, R^\alpha)(K), \alpha = 0, 1/2, 1, 3/2.... \tag{14}$$

as formal power series.

3.2 Some examples in the $SL(2, \mathbb{R})$ case

Let us now describe some infinite dimensional examples. Consider the Lie group $G = SL(2, \mathbb{R})$. It is a non-compact semisimple group. As before we have $\mathfrak{g}_{\mathbb{C}} = \mathfrak{sl}(2, \mathbb{C})$. Take again t to be the infinitesimal R-matrix coming from minus the Cartan-Killing form in $\mathfrak{g}_{\mathbb{C}}$. We can also write it as

$$t = -\frac{1}{4}\left(E \otimes F + F \otimes E + \frac{H \otimes H}{2}\right), \tag{15}$$

where:

$$H=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, E=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, F=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}. \tag{16}$$

We start by defining the representations of $SL(2,\mathbb{R})$ in the principal series. They depend on an imaginary parameter $s\in i\mathbb{R}$ and an $\varepsilon\in\{0,1\}$, the parity of the representation. In general $V^{s,\varepsilon}$ is the space $L^2(\mathbb{R})$ of complex-valued square integrable functions in $\mathbb{R}$. The action of $SL(2,\mathbb{R})$ has the form:

$$R^{s,\varepsilon}\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)(f)(x)=\mathrm{sgn}^{\varepsilon}(bx+d)|bx+d|^{s-1}f\left(\frac{ax+c}{bx+d}\right). \tag{17}$$

See [GI] for an alternative description of these representations, as well as the definition of their associated spin network theory. It applies to the construction of spin foam models for $(2+1)$-Quantum Gravity.

The positive discrete series depends on a parameter $m\in\mathbb{Z}^-$. The representations of this type are denoted by $R^{m,+}$. The representation space for the representation $\mathbb{R}^{m,+}$ is the space of holomorphic functions f in the upper half plane such that:

$$\frac{i}{2\Gamma(-m)}\int_{\mathrm{Im}(z)>0}|f(z)|^2(\mathrm{Im}(z))^{-m-1}dz d\bar{z}<+\infty. \tag{18}$$

The inner product in this Hilbert space has an expression similar to the formula above. The group $SL(2,\mathbb{R})$ acts in the fashion:

$$R^{s,\varepsilon}\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\right)(f)(x)=(bx+d)^{s-1}f\left(\frac{ax+c}{bx+d}\right). \tag{19}$$

The representations in the negative series are denoted by $R^{m,-}$. They depend on a parameter $m\in\mathbb{Z}^-$. The representation space for them is the space of holomorphic functions f in the lower half plane such that:

$$\frac{1}{2\Gamma(-m)}\int_{\mathrm{Im}(z)>0}|f(z)|^2|\mathrm{Im}(z)|^{-m-1}dz d\bar{z}<+\infty. \tag{20}$$

The action of $SL(2,\mathbb{R})$ in V is similar to the case of the positive discrete series.

The Lie group $SL(2,\mathbb{R})$ still has one more series of unitary representation, namely the complementary series of representations, and actually some more representations in the complementary series, now for $m > 0$. These last are defined as before but quotienting out invariant spaces of homogeneous polynomials out of them. Details can be found in [L] or [GGV]. Excluding the representation $R^{0,1}$, all the representations considered are unitary and irreducible. Therefore there is attached to them a knot invariant with values in $\mathbb{C}[[h]]$. In the next section we relate these knot invariants to the coloured Jones polynomial.

3.3 The z-Coloured Jones Polynomial as a universal invariant

Let G be a real Lie group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. We suppose $\mathfrak{g}_{\mathbb{C}}$ is equipped with a infinitesimal R-matrix $t \in \mathfrak{g}_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}}$. Let also R be a irreducible unitary representation of G in complex Hilbert space V. A closer look at the definition of the knot invariant $I(G,t,R)$, cf Theorem 22, tells us that the only information which we took from the representation R was its infinitesimal character χ_R^∞. It is a morphism of algebras from the centre $\mathcal{C}(U(\mathfrak{g}_{\mathbb{C}}))$ of $U(\mathfrak{g}_{\mathbb{C}})$ to $\mathbb{C}$, that is a central character of $U(\mathfrak{g}_{\mathbb{C}})$. In the case $\mathfrak{g}_{\mathbb{C}}$ is semisimple, we know the form of all such morphisms. Let us say what the situation is in the case $\mathfrak{g}_C = \mathfrak{sl}(2,\mathbb{C})$. We refer to [VAR], or to any classical reference on Lie algebras for further details.

The description we are going to give now generalises to any semisimple Lie algebra. In particular any similar construction of knot invariants out of infinite dimensional representations of semisimple Lie groups will have the same kind of properties.

Consider any Cartan decomposition of $\mathfrak{sl}(2,\mathbb{C})$ and choose a Borel subalgebra relative to it. Given a complex number a there exists a unique irreducible, cyclic, highest weight representation $\rho^{\frac{a}{2}}$ of maximal weight a. This representation is finite dimensional if, and only if, $a = 0, 1, 2, ...$ In this case it integrates to the representation of $SL(2,\mathbb{C})$ of spin $\frac{a}{2}$. If a is an arbitrary complex number, any element of the centre of $U(\mathfrak{sl}(2,\mathbb{C}))$ acts in the representation space of $\rho^{\frac{a}{2}}$ as a multiple of the identity. Denote by $\lambda_{\rho^{\frac{a}{2}}}$ the central character of $\rho^{\frac{a}{2}}$. That is if x is a central element of $U(\mathfrak{sl}(2,\mathbb{C}))$ then $\lambda_{\rho^{\frac{a}{2}}}(x)$ is the unique complex number such that $\lambda_{\rho^{\frac{a}{2}}}(x)v = \rho^{\frac{a}{2}}(x)(v)$ for all v in the representation space of $\rho^{\frac{a}{2}}$. We will then have $\lambda_{\rho^{\frac{a}{2}}} = \chi^{\infty}_{R^{\frac{a}{2}}}, a \in \{0, 1, 2, ..\}$. Recall R^{α} denotes the representation of $SU(2)$ of spin $\alpha = 0, 1/2, 1, 3/2....$ If we consider the usual Cartan decomposition of $\mathfrak{sl}(2,\mathbb{C})$ then the irreducible cyclic highest weight representation with highest weight $2z$ is exactly the representation $\tilde{\rho}^{z}$ of spin z defined in 1.4.6.

All the pieces of the following theorem can be found in [VAR], and almost all the classical references on Lie algebras.

Theorem 24 *Let $f : \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C}))) \to \mathbb{C}$ be a central character of $U(\mathfrak{sl}(2,\mathbb{C}))$. We have:*

1. *There exists an $a \in \mathbb{C}$ such that $f = \lambda_{\rho^{\frac{a}{2}}}$.*
2. *$\lambda_{\rho^{\frac{a}{2}}} = \lambda_{\rho^{\frac{b}{2}}}$ if and only if $(a+1)^2 = (b+1)^2$.*
3. *Given $x \in \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C})))$ the map $a \in \mathbb{C} \mapsto \lambda_{\rho^{\frac{a}{2}}}(x)$ is a polynomial in a of degree smaller or equal to the degree of x in $U(\mathfrak{sl}(2,\mathbb{C}))$. In fact it is a polynomial in $(a+1)^2$*

This theorem admits an obvious generalisation to any semisimple Lie algebra.

Notice that the z-coloured Jones polynomial is defined as the form: (cf Theorem 23)

$$\frac{J^z}{2z+1}(K) = \lambda_{\rho^z} \circ (\phi_t \circ \mathcal{Z})(K), \tag{21}$$

here K denotes a framed knot. Due to part 3. of Theorem 24 we can see again that:

$$\frac{J^z}{2z+1}(K) = \sum_{n=0}^{+\infty} J_n^z(K) h^n = \sum_{n=0}^{+\infty} \left(\sum_{k=0}^{2n} J_{n,k}(K) z^k \right) h^n \tag{22}$$

Where the degree of $(\phi_t \circ \mathcal{Z}_n)(K)$ in $U(\mathfrak{sl}(2, \mathbb{C})$ is not bigger than $2n$, for any framed knot K. Some other properties of the z-coloured Jones polynomial coming from the corresponding properties of the Melvin-Morton expansion are the following:

1. If K is a framed knot then $J_n^z(K)$ is a polynomial in $(2z+1)^2$ of degree smaller or equal to n.

2. If $(2z+1)^2 = (2w+1)^2$ then $\frac{J^z}{2z+1} = \frac{J^w}{2w+1}$.

3. $\frac{J^z}{2z+1}$ is the usual (rescaled) Jones polynomial if $2z+1 = 1, 2, ...$

4. If $2z+1 = 1, -1, 2, -2...$ then $\frac{J^z}{2z+1}(K)$ defines a power series in h convergent in $\mathbb{C}$.

Properties 1, 2 and 3 are easy consequences of our discussion. The fourth is a consequence of the fact that, in our normalisation, $\frac{J^z}{2z+1}(K)$ is a Laurent polynomial in $e^{h/4}$ if $2z+1 = 1, -1, 2, -2...$, as referred in 1.4.2. This generalises to any semisimple Lie algebra.

3.3.1 Some more examples of Melvin-Morton expansions

We now derive some Melvin-Morton expansions from known formulae of the Coloured Jones Polynomial of some simple knots. These formulae will be useful later.

Let $q = \exp(h)$. For an $n \in \mathbb{N}_0$ and a $z \in \mathbb{C}$, consider the term

$$D(n,z) = \prod_{k=1}^{n} \left[\left(q^{\frac{2z+1}{2}} - q^{-\frac{2z+1}{2}} \right)^2 - \left(q^{\frac{k}{2}} - q^{-\frac{k}{2}} \right)^2 \right]. \tag{23}$$

It is a power series in h such that the first $2n$ terms are zero. Therefore if $f(n), n \in \mathbb{N}_0$ are power series in h, for example Laurent polynomials in q and q^{-1}, then $\sum_{n\in\mathbb{N}_0} f(n)D(n,z)$ is an infinite series of power series which is termwise convergent, since it is of terminating type, termwise. Suppose $A(z) = \sum_{n\in\mathbb{N}_0} A_n(z)h^n$ and $B(z) = \sum_{n\in\mathbb{N}_0} B_n(z)h^n$ are power series whose coefficients depend polynomially in z, for example power series such as $q^{\frac{2z+1}{2}}$ or $q^{-\frac{2z+1}{2}}$. Then also the coefficients of their product depend polynomially in z, thus in particular the coefficients of $D(n,z)$, for any $n \in \mathbb{N}_0$. The same is true for the coefficients of any power series of the type $\sum_{n\in\mathbb{N}_0} f(n,z)D(n,z)$, where $f(n,z) \in \mathbb{C}[[h]]$ is such that its terms depend polynomially in z.

Let 3_1 and 4_1 denote the zero framed trefoil and figure of eight knots. We have, see [H]:

$$\frac{J^z}{2z+1}(3_1) = \frac{1}{2z+1} \frac{q^{\frac{2z+1}{2}} - q^{-\frac{2z+1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{n=0}^{+\infty} (-1)^n q^{-n(n+3)/2} D(n,z), \tag{24}$$

$$\frac{J^z}{2z+1}(4_1) = \frac{1}{2z+1} \frac{q^{\frac{2z+1}{2}} - q^{-\frac{2z+1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{n=0}^{+\infty} D(n,z). \tag{25}$$

In general for any framed knot K there exist Laurent polynomials $f_n(K)(h), n \in \mathbb{N}_0$

in q and q^{-1} such that

$$\frac{J^z}{2z+1}(K) = q^{F(K)z(z+1)} \frac{q^{\frac{2z+1}{2}} - q^{-\frac{2z+1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \sum_{n=0}^{+\infty} f_n(K) D(n,z), \tag{26}$$

where $F(K)$ is the framing coefficient of K. Actually, Habiro proved these formulae only in the case of finite dimensional representations, that for $z \in \frac{1}{2}\mathbb{N}_0$. However the coefficients of the power series above depend polynomially in z, which implies the formulae are true also for infinite dimensional representations. This is the old principle that if two polynomials coincide in an infinite set then they are the same. We have used this method to prove Theorem 16. We will use it again quite frequently, in fact in most of all major results we prove. Notice that equation 26 also proves that $\frac{J^\alpha}{2\alpha+1}(K)$ always defines a Laurent polynomial in $q^{1/4}$ if $\alpha = 0, \frac{1}{2}, 1, ...$

3.3.2 Back to $SL(2,\mathbb{R})$

Recall the framed knot invariants $I(G,t,R)$ defined in theorem 22. Given that $\mathfrak{sl}(2,\mathbb{C})$ is simple, we can prove that any infinitesimal R-matrix in $\mathfrak{sl}(2,\mathbb{C})$ is a multiple of the one coming from minus the Cartan-Killing form considered previously, see 1.3.1. The following result is a straightforward consequence of the discussion above (especially of Theorem 24):

Theorem 25 *Let G be a real form of $SL(2,\mathbb{C})$ and R be an irreducible unitary representation of G in the Hilbert space V. Let also t be an infinitesimal R-matrix in $\mathfrak{sl}(2,\mathbb{C})$. After rescaling t (possibly), there exists a $z \in \mathbb{C}$ such that $I(G,t,R) = \frac{J^z}{2z+1}$.*

We can prove similar results for any semisimple Lie group.

It is possible to find the exact relation between the invariants associated with the unitary representations of $SL(2,\mathbb{R})$ and the z-coloured Jones polynomial which needs to exist in the light of the theorem above. Let as usual t be the infinitesimal R-matrix in $\mathfrak{sl}(2,\mathbb{C})$ coming from minus its Cartan-Killing form. In general the highest weight representations $\rho^a, a \in \mathbb{C}$ of $\mathfrak{sl}(2,\mathbb{C})$ cannot be integrated to representations of $SL(2,\mathbb{R})$, however it is possible to relate their central characters with the infinitesimal characters of the unitary representations of $SL(2,\mathbb{R})$. We will do it in 5.3. which will prove:

Theorem 26 *Let t be the infinitesimal R-matrix in $\mathfrak{sl}(2,\mathbb{C})$ coming from minus its Cartan-Killing form. We have, for the principal series of representations:*

$$I(SL(2,\mathbb{R}), t, R^{s,\pm 1}) = \frac{J^{\frac{s-1}{2}}}{s}, s \in i\mathbb{R}, \tag{27}$$

and

$$I(SL(2,\mathbb{R}), t, R^{m,\pm}) = \frac{J^{\frac{m-1}{2}}}{m}, m \in \mathbb{Z}^{-}, \tag{28}$$

for the discrete series.

Observe that as a consequence the usual Coloured Jones Polynomial can be obtained out of the unitary infinite dimensional representations of $SL(2,\mathbb{R})$ in the discrete series.

Another consequence is the fact observed in the introduction that the quantum knot invariants associated with the infinite dimensional representations $SL(2,\mathbb{R})$ are in a sense analytic continuations of the ones associated with finite dimensional representations of the complexification $\mathfrak{sl}(2,\mathbb{C})$ of $\mathfrak{sl}(2,\mathbb{R})$. Also the fact that the non-compact knot invariants are not stronger than the ones associated with finite dimensional representations. The meaning of this is obvious from the Melvin-Morton expansion (22). Notice however that this property is valid only termwise in the power series expansions, and we will see later that in some cases the analytically continued power series may have a zero radius of convergence. This does not happen in the case of the Coloured Jones polynomial. We will come back to this in Chapter 4. This property concerning analytic continuations is valid if G is any semisimple Lie group. I do not know what the answer is in the case of non semisimple Lie algebras.

3.3.3 The Lorentz Polynomial

Recall 2.1. The Lie algebra $L \cong \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ is the complexification of the lie algebra of $\mathfrak{sl}(2,\mathbb{C})$ seen as a real Lie group. As before, consider the infinitesimal R-matrix t_L in $L_\mathbb{C}$ given by $t_L = t^l - t^r$. After theorem 16, given z and w in $\mathbb{C}$ it is thus natural to define the Lorentz polynomial as being:

$$\frac{L^{z,w}}{(2z+1)(2w+1)}(K) = \frac{J^z}{2z+1}(K^*)\frac{J^w}{2w+1}(K). \tag{29}$$

Here K is a framed knot and K^* denotes its mirror image.

Similarly with the $SL(2,\mathbb{R})$ case, we have:

Theorem 27 *Let R be a irreducible unitary representation of $SL(2,\mathbb{C})$ in the complex Hilbert space V. There exist $z, w \in \mathbb{C}$ such that:*

$$I(SL(2,\mathbb{C}), t_L, R) = \frac{L^{z,w}}{(2z+1)(2w+1)}. \tag{30}$$

Proof. Let $f = \chi_R^\infty : \mathcal{C}(L) \to \mathbb{C}$ be the infinitesimal character of R, thus

$$I(SL(2,\mathbb{C}), t_L, R) = f \circ \phi_{t_L} \circ \mathcal{Z}.$$

As we have seen in 1.3.4, if K is a framed knot we have

$$(\phi_{t_L} \circ \mathcal{Z})(K) = (\phi_t \circ \mathcal{Z})(K) \otimes (\phi_{-t} \circ \mathcal{Z})(K) \in U(\mathfrak{sl}(2,\mathbb{C})) \otimes U(\mathfrak{sl}(2,\mathbb{C})) \cong U(L).$$

The central character $f : \mathcal{C}(U(L)) \to \mathbb{C}$ is the central character of a irreducible, cyclic highest weight representation of $U(L)$. Such representation is the direct sum of two irreducible cyclic highest weight representations of $U(\mathfrak{sl}(2,\mathbb{C}))$ of maximal weights $2w$ and $2z$. They have central characters λ_{ρ^z} and λ_{ρ^w}, thus $f = \lambda_{\rho^z} \otimes \lambda_{\rho^z}$ under the isomorphism $\mathcal{C}(U(L)) \cong \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C}))) \otimes \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C})))$. In particular

$$f \circ \phi_{t_L} \circ \mathcal{Z} = (\lambda_{\rho^z} \circ \phi_t \circ \mathcal{Z}) \otimes (\lambda_{\rho^w} \circ \phi_{-t} \circ \mathcal{Z}) = \frac{L^{z,w}}{(2z+1)(2w+1)},$$

for some z and w in $\mathbb{C}$. ■

As an example let us consider the principal series of unitary representations of $SL(2,\mathbb{C})$. Their infinitesimal counterpart was defined in 2.1.1. We want to describe the actual group representations now. For a more geometric description these representations in terms of hyperbolic geometry we refer to [GGV]. The unitary principal series is parametrised by a pair of complex numbers z and w with $m = z - w \in \mathbb{Z}$ and $i\rho = z + w + 1 \in i\mathbb{R}$. In general the parameters m and ρ are referred to as the minimal spin and the mass of the representation. The ones of minimal spin 0 are the balanced representations of [BC]. As observed in the same reference, they admit a natural spin network theory. The representation space V for $R^{m,\rho}$ is $L^2(\mathbb{C})$ and the action of $SL(2,\mathbb{C})$ has the form:

$$R^{z,w}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(f)(\xi) = (b\xi + d)^{z-1}(\bar{b}\bar{\xi} + \bar{d})^{w-1} f\left(\frac{a\xi + c}{b\xi + d}\right). \tag{31}$$

The representations in the principal series are unitary and irreducible, therefore there exists a knot invariant attached to them. As before, these knot invariants are particular cases of the Lorentz polynomial. Since, as referred to in 1.4.6 the infinitesimal character of $R(m,p)$ is $\rho(m,p)$, we have by Theorem 16:

Theorem 28 *Let $\rho = z + w + 1$ and $m = z - w$ we have*

$$I(SL(2,\mathbb{C}), t_L, R^{m,\rho}) = \frac{L^{z,w}}{(2z+1)(2w+1)} = X(m,p). \tag{32}$$

4 Convergence issues

We now look at the analytic properties of the z-coloured Jones polynomial. As we pointed out for several times before, the power series coming out of it are in general not convergent (that is they have a zero radius of convergence). We now show this is what happens at least in the case of torus knots. Later we will have a look at the properties of the z-coloured Jones polynomial under Borel re-summation. This is an expanded version of the corresponding chapter of [FM1].

4.1 On the divergence of the z-Coloured Jones Polynomial power series for torus knots

Let m and p be two coprime positive integers. In what follows $K_{m,p}$ denotes the (m,p)-Torus Knot, see [Wi1] for example. For each $z \in \mathbb{C}$, consider the following meromorphic function:

$$F_{m,p,z}(x) = \frac{\sinh\left((2z+1)\sqrt{mp}x\right)\sinh\left(\sqrt{\frac{m}{p}}x\right)\sinh\left(\sqrt{\frac{p}{m}}x\right)}{(2z+1)\sinh\left(\sqrt{mp}x\right)}. \tag{33}$$

Notice it is well defined if $2z+1=0$. It is an even function in x. Suppose $2z+1 \in \mathbb{Z}\setminus\{0\}$, it is possible to prove that: see [KT]

$$\frac{J^z}{2z+1}(K_{m,p})(h) = \frac{1}{2\sqrt{\pi}}\frac{e^{-\frac{h}{4}\left(\frac{p}{m}+\frac{m}{p}\right)}}{\sinh\left(\frac{h}{2}\right)}\int_{-\infty}^{+\infty} e^{-x^2}F_{m,p,z}(\sqrt{h}x)dx, \forall h \in \mathbb{C}. \tag{34}$$

This is a consequence of a previous formula proved in [Mor]. Notice that for $2z+1 \in \mathbb{Z}\setminus\{0\}$ the power series $\frac{J^z}{2z+1}(K_{m,p})(h)$ has an infinite radius of convergence. The origin is never a singular point of $F_{m,p,z}(x)$ for any $z \in \mathbb{C}$, thus if $\sqrt{h}x$ is small enough:

$$F_{m,p,z}(\sqrt{h}x) = \sum_{k=1}^{+\infty} Q_{m,p,z}(k) h^k x^{2k}. \tag{35}$$

In the case $2z+1 \in \mathbb{Z}\setminus\{0\}$ the function $F_{m,p,z}(x)$ is an entire function of exponential order. That is f is analytic in $\mathbb{C}$ and there exist positive constants A and C such that $|F_{m,p,z}(x)| < Ae^{C|x|}, \forall x \in \mathbb{C}$. See 5.4.1. This implies that we have the bound $|Q_{m,p,z}(k)| \leq D^{2k}/(2k)!, k \in \{1, 2, ...\}$ for some $D < \infty$, by Theorem 52. In practice this means that, for any $h \in \mathbb{C}$, we can interchange the infinite summation with the integral sign in the following expression:

$$\frac{J^z}{2z+1}(K_{m,p})(h) = \frac{1}{2\sqrt{\pi}} \frac{e^{-\frac{h}{4}\left(\frac{p}{m}+\frac{m}{p}\right)}}{\sinh\left(\frac{h}{2}\right)} \int_{-\infty}^{+\infty} e^{-x^2} \left(\sum_{k=1}^{+\infty} Q_{m,p,z}(k) h^k x^{2k} \right) dx, \forall h \in \mathbb{C}, \tag{36}$$

valid if $2z+1 \in \mathbb{Z}\setminus\{0\}$. This is because all partial sums are bounded by the function $\exp(-x^2)\exp(D|x|\sqrt{|h|})$, which is integrable. Therefore:

$$\frac{J^z}{2z+1}(K_{m,p})(h) = \frac{1}{2\sqrt{\pi}} \frac{e^{-\frac{h}{4}\left(\frac{p}{m}+\frac{m}{p}\right)}}{\sinh\left(\frac{h}{2}\right)} \sum_{k=1}^{+\infty} \Gamma(k+\frac{1}{2}) Q_{m,p,z}(k) h^k, \tag{37}$$

if $2z+1 \in \mathbb{Z} \setminus \{0\}$ and $h \in \mathbb{C}$. Given that the coefficients of the Taylor decomposition of $\sinh((2z+1)\sqrt{mp}x)$ at $x = 0$ depend polynomially in z we show that the dependence of the coefficients $Q_{m,p,z}(k)$ in z is in fact polynomial. Thus the expansion (37) is true for any $z \in \mathbb{C}$, now only at the power series level. We have shown:

Theorem 29 *Let m and p be coprime positive integers and $K_{m,p}$ be the (m,p)-torus knot. The expansion (37) of the z-coloured Jones polynomial of $K_{m,p}$ is correct for any $z \in \mathbb{C}$, as formal power series*

The functions $F_{m,p,z}(x)$ have non removable singularities if $2z+1 \notin \mathbb{Z} \setminus \{0\}$. In particular there exists a positive constant C such that $Q_{m,p,z}(k) > C^k$ for infinite k's. As a consequence we can conclude:

Corollary 30 *Let $K_{m,p}$ be the m,n-torus knot. The power series $\frac{J^z}{2z+1}(K_{m,p})$ has a zero radius of convergence if $2z+1 \notin \mathbb{Z} \setminus \{0\}$.*

It is easy to conclude that the result above is true also for the mirror images of the class of torus knots just considered.

It is possible to prove a similar result of divergence in the Lorentz group case: Let $z, w \in \mathbb{C}$. Consider the function

$$G_{m,p,z,w}(x) = \frac{1}{4\pi} \int_0^{2\pi} F_{m,p,z}(x\cos(\theta)) F_{m,p,w}(ix\sin(\theta)) d\theta, \tag{38}$$

thus if $2z+1, 2w+1 \in \mathbb{Z} \setminus \{0\}$ we have:

$$\frac{L^{z,w}}{(2z+1)(2w+1)} = \frac{1}{\sinh\left(\frac{-h}{2}\right)\sinh\left(\frac{h}{2}\right)} \int_0^{+\infty} e^{-x^2} x G_{m,p,z,w}(x\sqrt{h}) dx, \tag{39}$$

from (34) If $2z+1$ and $2w+1$ are non zero integers, then $G_{m,p,z,w}$ is entire of exponential order. Otherwise it has non removable singularities. For, suppose, for example, that $2w+1 \notin \mathbb{Z} \setminus \{0\}$, let w_0 be the first singularity of $F_{m,p,w}(ix)$ in the positive real line. It is a pole. An explicit calculation tells us that as x approaches w_0 from below along the real line then the first derivative of $G_{m,p,z,w}(x)$ tends to ∞.

Given any $z, w \in \mathbb{C}$, the function $G_{m,p,z,w}(x)$ is an even function with a zero of order 4 at the origin. Put

$$G_{m,p,z,w}(x) = \sum_{k=2}^{\infty} P_{m,p,z,w}(k) x^{2k}. \tag{40}$$

For fixed k, the dependence of $P_{m,p,z,w}(k)$ in z and w is polynomial. In fact, if we fix m, p, z, w $(z, w \in \mathbb{C}$ then for x small enough the series

$$F_{m,p,x}(x\cos\theta)F_{m,p,w}(ix\sin(\theta)) = \sum_{a,b=0}^{\infty} Q_{m,p,z}(a)Q_{m,p,w}(b)x^{a+b}i^b\cos^a(\theta)\sin^b(\theta) \quad (41)$$

converges uniformly for $\theta \in [0, 2\pi]$. Thus

$$G_{m,p,z,w}(x) = \sum_{a,b} C_{a,b}Q_{m,p,z}(a)Q_{m,p,w}(b)x^{a+b} \quad (42)$$

where

$$C_{a,b} = \int_0^{2\pi} i^b \cos^a(\theta)\sin^b(\theta)d\theta. \quad (43)$$

This proves $P_{m,p,z,w}(k)$ is a polynomial in z and w for any k. Since $G_{m,p,z,w}(x)$ is entire of exponential order if $2z+1, 2w+1 \in \mathbb{Z}\setminus\{0\}$, similarly as above, we conclude:

$$\frac{L^{z,w}}{(2z+1)(2w+1)} = \frac{1}{2\sinh\left(\frac{h}{2}\right)\sinh\left(\frac{-h}{2}\right)}\sum_{k=1}^{+\infty} P_{m,p,z,w}(k)k!h^{k+1}, \quad (44)$$

from which follows the non-convergence of the power series defined by the Lorentz polynomial if $2z+1$ or $2w+1$ do not belong to $\mathbb{Z}\setminus\{0\}$, in the case of torus knots.

We are mainly interested in knot invariants with values in $\mathbb{C}$. As before we refer to them as numerical knot invariants. It is natural now to ask whether we can canonically find analytic functions of which the z-coloured Jones polynomial in a knot are asymptotic developments. In the remainder of this chapter we investigate this question in the framework of Borel re-summation.

Observation 31 *Professor Hugh Morton suggested to me the results by Rozansky which were conjectured in [R1] and proved in [R2] may provide the framework for a general proof of the divergence of the z-coloured Jones polynomial, for the case of knots with a trivial Alexander polynomial.*

4.2 Borel Re-summation of power series

We make now a brief description of the Borel process of re-summation of power series. We refer to [SS] for full details and for proofs (which sometimes are not that simple). The paper [FL] contains a simple and illuminating introduction to this subject.

4.2.1 Asymptotic power series developments and a lemma due to Borel

Consider a sector $\Omega = \{z \in \mathbb{C} : \theta_1 \leq \arg(z) \leq \theta_2, |z| \leq C\}$ in the complex plane (or in the Riemann surface of the Logarithm, if $\theta_2 - \theta_1 \geq 2\pi$). Let $\sum_{n=0}^{+\infty} a_n h^n$ be a formal power series which we do not assume to be convergent. Suppose $f : \Omega \to \mathbb{C}$ is a continuous function in Ω and analytic in the interior $\mathrm{int}(\Omega)$ of Ω. We say that $\sum_{n=0}^{+\infty} a_n h^n$ is an asymptotic expansion of f at the origin if, for any proper subsector Ω' of Ω, and any $N = 1, 2...$, there exists a positive constant C_N such that

$$\left| f(h) - \sum_{n=0}^{N} a_n h^n \right| \leq C_N |h|^{N+1}, \forall h \in \Omega'. \tag{45}$$

For example the, horribly divergent, series $\sum_{n=0}^{+\infty} (-1)^n (n-1)! h^n$ is an asymptotic series for the, perfectly well defined, function $h \mapsto \int_0^{+\infty} e^{-\frac{x}{h}} . \frac{1}{x+1} dx$ in any sector $\Omega = \{h \in \mathbb{C} : \mathrm{Re}(h) \geq 0, |h| \leq C\}$. Here C is a positive constant.

Let Ω be a sector in the complex plane. Denote by $G(\Omega)$ the space of holomorphic function in $\mathrm{int}(\Omega)$ and continuous in Ω which admit a power series asymptotic expansion at the origin. It can be proved that $G(\Omega)$ is closed under differentiation and it is an algebra with respect to multiplication of complex functions. If $\sum_{n=0}^{+\infty} a_n h^n$ is a asymptotic power series expansion of some $f \in G(\Omega)$ then the coefficients a_n are unique and can be computed by the formula:

$$a_n = \lim_{\substack{h \to 0 \\ h \in \Omega}} \frac{1}{n!} f^{(n)}(h). \tag{46}$$

This permits us to define a map $As : G(\Omega) \to \mathbb{C}[[h]]$. It can be proved that As is an algebra morphism and it is well behaved with respect to differentiation. Moreover it is a surjective map and its kernel is the space of functions analytic in $\mathrm{int}(\Omega)$ and continuous in Ω that tend to zero faster than any z^n for any $n \in \mathbb{N}$. This fact is known as Borel's lemma. It permits us to define a re-summation operator of formal powers series up to functions that go to zero faster that any natural power of z as $z \to 0$.

In practice, to re-sum a power series means finding an analytic function this power series is an asymptotic expansion of. Therefore Borel's lemma tells us that we can always re-sum them. For practical applications, however, it is important to reduce as much as possible the uncertainty in the process of re-summation. This is the subject of the following paragraph.

4.2.2 Power series in the First Gevrey Class and Formal Borel Transforms

Let $\sum_{n=0}^{+\infty} a_n h^n$ be a formal power series. We say that it is of type Gevrey 1 if there exists a positive constant C such that $|a_n| \leq C^n n!, \forall n \in \mathbb{N}$. We denote by $G_1[[h]]$ the algebra of formal power series of type Gevrey 1. If Ω is a sector in the complex plane, then a function in $G(\Omega)$ is said to be of type Gevrey 1 if its asymptotic expansion at zero belongs to $G_1[[h]]$. We also impose that for any subsector Ω' of Ω

there exists a positive C such that

$$\left|\frac{f^{(n)}(z)}{n!}\right| \leq C^n n!, \forall z \in \Omega', \forall n \in \mathbb{N}.$$

It is easy to prove that the first condition follows from this last one. Notice that $G_1(\Omega)$ is an algebra stable under differentiation.

Suppose the opening of the sector Ω is less than π. Then as before $As_1 : G_1(\Omega) \to G_1[[h]]$ is a surjective algebra morphism, and its kernel is the space of exponential decreasing functions in Ω. That is, functions that satisfy the estimate $f(h) \leq Ae^{-\frac{B}{|h|}}$ for some positive A and B in any proper subsector of Ω.

Recall that the Borel transform $\mathcal{B}(f)$ of an analytic function f, if it exists, is the inverse Laplace transform of it. We refer again to [SS] for a discussion of this subject in the generality we need. It is well known that $\mathcal{B}(1) = \delta(\xi)$ and $\mathcal{B}(h^{n+1}) = \xi^n/n!$. Notice we do a change of variables $h \mapsto 1/h$ in the domain of the Laplace transform.

Let $\sum_n a_n h^n$ be a formal power series in the first Gevrey class. Consider the generalised function

$$F(\xi) = \mathcal{B}(\sum_{n=0}^{+\infty} a_n h^n)(\xi) = \sum_{n=0}^{+\infty} \frac{a_{n+1}}{n!}\xi^n + a_0\delta(\xi). \tag{47}$$

It is called the Formal Borel transform of $\sum_{n=0}^{+\infty} a_n h^n$. Let $0 < a < C$ be a real number. Choose a direction $e^{i\theta}$ in the complex plane such that $\mathrm{Re}(e^{i\theta}/h) > 0, \forall h \in \Omega$. Recall Ω opens less than π. It is possible to prove that the incomplete Laplace Transform in the direction $e^{i\theta}$:

$$h \mapsto \mathcal{L}^a_{e^{i\theta}}(F(\xi))(h) = \int_0^{ae^{i\theta}} e^{\frac{-\xi}{h}} F(\xi) d\xi \tag{48}$$

has $\sum_{n=0}^{+\infty} a_n h^n$ as a power series asymptotic development at the origin. This defines a re-summation of power series of Gevrey type 1, up to exponentially decreasing functions.

4.2.3 Re-summation Operators

For a better definition of the re-summation operator, we are led to consider formal power series whose Formal Borel transform can be analytically continued to the neighbourhood of some ray $e^{i\theta}\mathbb{R}_0^+$ in the complex plane, cf [SS]. This will lead to the definition of a re-summation operator up to rapidly decreasing functions. That is, functions that verify $\forall A > 0, \exists B > 0 : |f(h)| \leq Be^{-A/|h|}, \forall h \in \Omega'$, in any proper subsector Ω' of Ω. In addition it will extend the usual sum of convergente formal power series.

Let $F(\xi)$ be an analytic function possibly containing a $\delta(\xi)$ term, in an open subset of $\mathbb{C}$ containing some ray $e^{i\theta}\mathbb{R}_0^+$. Suppose it grows at most exponentially along it. The Laplace transform of $F(\xi)$ in the direction $e^{i\theta}$ is defined as:

$$\mathcal{L}_{e^{i\theta}}(F(\xi))(h) = \int_0^{+\infty e^{i\theta}} e^{-\xi/h} F(\xi) d\xi, \tag{49}$$

whenever the integral is convergent. A lot of properties of this changed Laplace transform can be deduced from the corresponding ones of the usual Laplace Transform. In particular $\mathcal{L}_{e^{i\theta}}(\xi^n) = \Gamma(n+1)h^{n+1}, n = 0, 1, ...$ If a is a real number bigger than -1 this generalises to $\mathcal{L}_1(\xi^a) = \Gamma(a+1)h^{a+1}$.

Definition 32 *(***re-summation operators***) Let $e^{i\theta}$ be a direction in the complex plane and $\sum_{n=0}^{+\infty} a_n h^n$ a formal power series of Gevrey type 1. We say it is $e^{i\theta}$-summable if its Formal Borel transform can be analytically continued to the neighbourhood of some ray $e^{i\theta}\mathbb{R}_0^+$, and if its Laplace transform converges in some nonempty subset of $\mathbb{C}$. If h belongs to the domain of definition of the Laplace trans-*

form then the $e^{i\theta}$ re-summation operator is defined as:

$$S\left(e^{i\theta},\sum_{n=0}^{+\infty}a_nh^n\right)=\mathcal{L}_{e^{i\theta}}\left(\mathcal{B}(\sum_{n=0}^{+\infty}a_nh^n)(\xi)\right)(h). \tag{50}$$

We can obviously define the re-summation operators along any curve that tends to the point at infinity. This re-summation operators are well behaved with respect to the multiplication of power series. Therefore, to avoid working with generalised functions, we redefine the re-summation operators and consider:

$$\mathcal{S}\left(e^{i\theta},\sum_{n=0}^{+\infty}a_nh^n\right)=\frac{1}{h}S\left(e^{i\theta},\sum_{n=0}^{+\infty}a_nh^{n+1}\right). \tag{51}$$

Definition 33 (Borel re-summability) *A formal power series $\sum_n a_nh^n$ is said to be Borel re-summable if the re-summation operators $\mathcal{S}$ make sense when applied to it.*

4.3 Back to Knots!

Let K be any framed knot. Consider a unitary representation R of $SL(2,\mathbb{R})$. It is possible to obtain some estimates for the coefficients of the Kontsevich Integral as well as for the matrix elements of R^∞, which are suitable to prove that $I(SL(2,\mathbb{R}),t,R)$ defines a formal power series of Gevrey type 1. This is the subject of Chapter 6. In fact:

Theorem 34 *Let K be a framed Knot, z and w two complex numbers. The series defined by $\frac{J^z}{2z+1}(K)$ and $\frac{L^{z,w}}{(2z+1)(2w+1)}(K)$ are of Gevrey type 1. That is, their Formal Borel Transform defines an analytic function in a neighbourhood of zero.*

This is probably the most difficult result of this thesis.

Notice equation (37) and Theorem 29 prove this for torus knots since the series $\sum_k Q_{m,p,z}(k)h^k$ has a positive radius of convergence. Recall the set of power series of Gevrey type 1 is a subalgebra of $\mathbb{C}[[h]]$.

We will give a full proof of this theorem in Chapter 6. It is a purely technical proof and the rest of the thesis is fairly independent of it. In 5.4.2 we prove this theorem for the Figure of Eight Knot and "sketch" a possible simple general proof of this result. We give a condensed proof of it in 5.5, a section extracted from [FM1].

4.3.1 The case of torus knots

We now analyse in detail the properties of the Formal Borel Transform of $\frac{J^z}{2z+1}(K)$ in the case in which K is a torus knot. We shall see the re-summation operators make sense in this case. The ambiguity in the process of re-summation is reduced to a finite set of functions differing by rapidly decreasing functions. The main idea is that equation (34) is almost a Laplace transform. In fact if we make the change of variables $x \mapsto \sqrt{x}/\sqrt{h}$ we can write it has:

$$\frac{J^z}{2z+1}(K_{m,p})(h) = \frac{1}{4\sqrt{\pi}} \frac{e^{-\frac{h}{4}\left(\frac{p}{m}+\frac{m}{p}\right)}}{\sinh\left(\frac{h}{2}\right)} \frac{1}{\sqrt{h}} \int_0^{+\infty} e^{-x/h} \frac{F_{m,p,z}}{2\sqrt{x}}(\sqrt{x})dx, \forall h \in \mathbb{R}^+. \quad (52)$$

Compare with (49). Notice that the right hand side of this equation also makes sense if $z \in \mathbb{C}$ and h is small enough. This will actually be our re-summation if $h \in \mathbb{R}^+$. Obvioully, some details are in need:

Let f and g be two complex valued functions continuous in some open subset of $\mathbb{C}$ which contains 0. Recall that their convolution is defined as

$$(f * g)(x) = \int_0^x f(x-\xi)g(\xi)d\xi, \quad (53)$$

whenever it makes sense. The contour of integration is chosen to be the segment connecting 0 and x. If f and g are entire then their convolution is an entire function. It is well know that if f and g grow at most exponentially along a ray $e^{i\theta}$ then the same is true for their convolution, and in addition $\mathcal{L}_{i\theta}(f*g)(h) = \mathcal{L}_{i\theta}(f)(h)\mathcal{L}_{i\theta}(g)(h)$ whenever all transforms makes sense. Suppose f and g are meromorphic functions which do not have the origin as a singular point. Let $D(f,g)$ be the set of points in $\mathbb{C}$ that can be connected to the origin by a straight line which does not pass by any singularity of f or g. The convolution of f and g is an analytic function in $D(f,g)$. The singularities of $f*g$ are in general ramifying due to the fact f and g may have non zero residua at their singular points. One can easily determine the Taylor series of $f*g$ in zero from the Taylor series of f and g, in fact if $f(x) = \sum_{n=0}^{+\infty} f_n x^n$ and $g(x) = \sum_{n=0}^{\infty} g_n x^n$ then $(f*g)(x) = \sum_{n,m=0}^{\infty} \frac{m!n!}{(m+n+1)!} f_m g_n x^{m+n+1}$, which converges whenever the Taylor series of f and g converge. This is because, in general, $x^m * x^n = \frac{m!n!}{(m+n+1)!}x^{m+n+1}; m,n \in \mathbb{N}$, apart from some simple analysis, basically the principles that a power series converges uniformly in any compact set contained in its disk of convergence, this disk being the maximal disk that does not contain any singularity. In particular if the coefficients of the Taylor series of $f_z(x)$ and $g_z(x)$ at zero depend polynomially in z then the same is true for the coefficients of the Taylor series of $f*g$. This fact will have prime importance in the sequel.

Suppose f is an odd meromorphic function non singular at the origin. Then the convolution $1/\sqrt{x} * f(\sqrt{x})$ defines an analytic function in $D(f)$, where $D(f)$ is the set of points in the complex plane that can be connected to zero by a straight line whose square does not meet any singularity of f. Notice $D(f)$ contains a neighbourhood of 0. In fact if $f(x) = \sum_{n=0}^{+\infty} a_n x^{2n+1}$ in a neighbourhood of the origin then $1/\sqrt{x} * f(\sqrt{x}) = \sum_{n=1}^{\infty} \frac{\Gamma(1/2)\Gamma(n+1+1/2)}{\Gamma(n+2)} a_n x^n$, in the same neighbourhood.

As before, the singular points of $1/\sqrt{x} * f(\sqrt{x})$ are usually ramifying due to the residua of f. Consider

$$I_{m,p,z} = \frac{\Gamma(\frac{1}{2})^{-1}}{\sqrt{x}} * \frac{F_{m,p,z}(\sqrt{x})}{2\sqrt{x}}. \tag{54}$$

Thus if $2z+1 \in \mathbb{Z} \setminus \{0\}$ then $I_{m,p,z}(x)$ is an entire function of exponential order. Moreover, by (52) we have

$$\frac{1}{2\sqrt{\pi}} \frac{e^{-\frac{h}{4}\left(\frac{p}{m}+\frac{m}{p}\right)}}{\sinh\left(\frac{h}{2}\right)} \mathcal{L}_1(I_{m,p,z}(x))(h) = \frac{J^z}{2z+1}(K_{m,p})(h), h > 0. \tag{55}$$

In general for any $z \in \mathbb{C}$, the function $I_{m,p,z}(x)$ is meromorphic, but it is well defined in a neighbourhood of zero. In addition the Taylor coefficients of $I_{m,p,z}$ depend polynomially in z, since the same is true for the Taylor coefficients of $F_{m,p,z}(x)$ in 0, as we have seen before. Let

$$H_{m,p,z}(x) = \mathcal{B}\left(\frac{h}{2\sqrt{\pi}} \frac{e^{-\frac{h}{4}\left(\frac{p}{m}+\frac{m}{p}\right)}}{\sinh\left(\frac{h}{2}\right)}\right). \tag{56}$$

It is an entire function of x. In fact it is a function of exponential order, the type (see 5.4.1) of which is smaller than $1/\pi + \varepsilon$, where $\varepsilon > 0$ is arbitrary. We can see it from the fact its Taylor coefficients are given by $\frac{1}{(n-1)!}a_n$ where $\sum_n a_n h^n$ has a radius of convergence equal to π. The Laplace transform in any direction always exists for h small enough. It extends analytically back the original function. This follows from the known principle that we can take the Laplace transform term by term in the power series developments of exponential order functions. We can prove it from the fact $|f^{(n)}(0)| \leq C^n, \forall n$, see 5.4.1.

Suppose m and p be coprime positive integers and let $K_{m,p}$ denote the (m,p)-Torus knot. From equation (55), it is immediate that if $2z+1 \in \mathbb{Z} \setminus \{0\}$ we have

$$h\frac{J^z}{2z+1}(K_{m,p}))(h) = \mathcal{L}_1\Big(H_{m,p,z}(x) * I_{m,p,z}(x)\Big)(h), h \in (0, 1/\pi) \tag{57}$$

Notice that $H_{m,p,z}(x)$ is exponential of type $1/\pi$. Thus by the unicity theorem for Laplace transforms it follows that

$$\mathcal{B}\left(h\frac{J^z}{2z+1}(K_{m,p})\right)(x) = H_{m,p,z}(x) * I_{m,p,z}(x), \forall x \in \mathbb{C}, 2z+1 \in \mathbb{Z} \setminus \{0\}. \quad (58)$$

We also need to use the fact that we can take the Laplace transform term by term in the Taylor expansion of $I_{m,p,z}(x)$ at zero to prove this. But this follows from the fact this function is of exponential order if $2z+1 \in \mathbb{Z} \setminus \{0\}$.

Now $H_{m,p,z}(x) * I_{m,p,z}(x)$ is a z-dependent family of analytic functions in a neighbourhood of zero, whose Taylor series coefficients in zero depend polynomially in z. In particular, equation (58) is true in the power series level. Therefore:

Theorem 35 *Let m, p be coprime integer and $K_{m,p}$ the (m,p)-torus Knot. The formal Borel tranform of $\frac{J^z}{2z+1}(K_{m,p}))$ extends analytically to the function $H_{m,p,z}(x) * I_{m,p,z}(x)$, well defined in $\mathbb{C} \setminus (-\infty, \pi^2/(mp)]$.*

Suppose $2z+1$ is not a non zero integer. Then the singularity of $H_{m,p,z}(x) * I_{m,p,z}(x)$ in $-\pi^2/(mp)$ is ramifying, since the singularity of $F_{m,p,z}$ in $i\pi/\sqrt{mp}$ has an non zero residue. Therefore we cannot remove it. This tells us that we cannot refine the estimate in Theorem 34.

In addition we can say that $H_{m,p,z}(x) * I_{m,p,z}(x)$ grows at most exponentially along any ray different of $\mathbb{R}^-$. Since the type of $H_{m,p,z}(x)$ as an exponential function is $1/\pi$, and $I_{m,p,z}(x)$ grows slower than $Ae^{\varepsilon|z|}$ for any $\varepsilon > 0$, this rate of exponential growth of $H_{m,p,z}(x) * I_{m,p,z}(x)$ can be taken smaller than $1/\pi + \varepsilon$ for any $\varepsilon > 0$.

We summarise our discussion and add some more simple facts:

Theorem 36 *Let m and p be coprime positive integers and $K_{m,p}$ be the (m,p)-Torus Knot. Let z be a complex number such that $2z+1 \notin \mathbb{Z} \setminus \{0\}$*

1. *The Formal Borel transform of $h^{J^z}_{2z+1}(K_{m,p})$ extends to an analytic function in $\mathbb{C} \setminus (-\infty, \pi^2/(mp)]$*

2. *The singularity in $-\pi^2/(mp)$ is a ramification point*

3. *If $e^{i\theta}\mathbb{R}_0^+$ is a ray in the complex plane that is not the set of non-positive real numbers then $\mathcal{B}(h^{J^z}_{2z+1}(K_{m,p}))(x)$ grows not faster than $Ae^{|x|/(\pi-\varepsilon)}$ along it where $\varepsilon > 0$ is arbitrary. This constant A can be chosen independent of the direction chosen.*

It is now very easy to analyse the re-summation of the z-coloured Jones polynomial for torus knots. Recall definition 32 and the comments after. Let $B(\pi) = \{h \in \mathbb{C}, |h| < \pi$. If $h \in B(\pi)$ consider $D(h) = \{w \in S^1 \setminus \{-1\} : \mathrm{Re}(w/h) > 1/\pi\} \neq \emptyset$. Given $h \in B(\pi)$ then $D(h)$ is connected if $\mathrm{Re}(h) \geq 0$ and it has two connected components otherwise.

Suppose $\mathrm{Re}(h) \geq 0$. Given that $D(h)$ only has one connected component, the re-summation $\mathcal{S}(w, h^{J^z}_{2z+1}(K_{m,p})(h))$ does not depend on $w \in D(h)$. This follows from point 3 of the previous theorem and Cauchy's integral formula. In addition, if $2z+1$ is a real or an imaginary number, the re-summation takes positive real values of h to real numbers, this is because the Borel transform is real in this case. Notice the last case corresponds to the representations of $SL(2,\mathbb{R})$ in the principal series.

Due to the fact that $D(x)$ has two connected components if $\mathrm{Re}(x) < 0$ the same is not true in this case. In fact if x is a real number smaller than 0 the two different re-summations will give conjugated complex numbers of non zero imaginary part for the case $2z+1$ is real or imaginary. This is obvious from the fact the Borel transform is real. Notice that the values of the re-summation, as well as the possible ambiguities can be read directly from equation (34).

The case of the mirror images of torus knots considered can be treated in a similar way. We just need to take the mirror image in the previous description. Therefore, for these knots the domain of re-summation is also $B(\pi)$. The re-summation procedure having 2 branches if $\mathrm{Re}(h) > 0$ and one otherwise. As before, the real parts of the re-summation for real h, are independent of the re-summation procedure if we stick to representations in the principal series.

The Lorentz group case can be dealt with considering the fact $\mathcal{L}_1(f(x)')(h) = 1/h\mathcal{L}_1(f(x))(h) - f(0)$, thus

$$\begin{aligned}&\mathcal{B}\left(\frac{h\mathcal{L}^{z,w}}{(2z+1)(2w+1)}(K_{m,p})\right)(x)\\&= \mathcal{B}\left(\frac{hJ^z}{(2z+1)}(K^*_{m,p})\right)(x) * \frac{d}{dx}\mathcal{B}\left(\frac{hJ^w}{(2w+1)}(K_{m,p})\right)(x)\end{aligned}$$

This is thus a well defined function in $\mathbb{C} \setminus ((-\infty, -\pi^2/(mp)] \cup [\pi^2/(mp), +\infty))$. It grows exponentially along any ray not contained in $\mathbb{R}$. Therefore, the domain for the re-summation of $\frac{L^{w,z}}{(2z+1)(2w+1)}(K_{m,p})$ is again $B(\pi)$. Unless h is imaginary, we now have always two branches for re-summation, if $2z+1$ and $2w+1$ are not non zero integers. Therefore

Theorem 37 *Let $K_{m,p}$ be the (m,p)-torus knot. For any $z, w \in \mathbb{C}$ the power series $\frac{J^z}{2z+1}(K_{m,p})(h)$ and $\frac{L^{z,w}}{(2z+1)(2w+1)}(K_{m,p})(h)$ are Borel re-summable.*

Recall that the process of Borel re-summation is well behaved with respect to the multiplication of power series. Therefore we can calculate the re-summation of the Lorentz Polynomial, including its ambiguities, out of the re-summation of the z-Coloured Jones Polynomial.

4.4 Conclusion to chapters 1, 2, 3 and 4

Let G be any real Lie group. As we have seen, if we are given an ad-invariant non degenerate symmetric bilinear form in $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, there exists a framed knot invariant $\phi_t \circ \mathcal{Z}$ with values in the algebra of formal powers series in h over the centre of the universal enveloping algebra of $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, in other words is is a formal power series of differential operators in G. If $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ is a semisimple Lie algebra and V is a finite dimensional representation of G then this formal power series at a particular value of h evaluates to an operator, or h-dependent family of operators, $V \to V$, which is a G-intertwiner. The trace of these operators will yield the usual quantum group knot invariants.

Instead of trying to make sense of this power series of differential operators at a particular value of h for the case V is an infinite dimensional representation, it is observed that each term of this power series acts as a multiple of the identity operator in the space of smooth vectors of any unitary irreducible representation. Therefore $\phi_t \circ \mathcal{Z}$ can be evaluated in any representation of this type. This gives us a knot invariant with values in the algebra of formal power series over $\mathbb{C}$ for any unitary irreducible representation of G. At least in the case $\mathfrak{g}$ is semisimple these knot invariants express out of the knot invariants associated with finite dimensional representations of the quantised universal enveloping algebra $U_h(\mathfrak{g})$ by a analytic continuation in the power series level, see theorem 26 and the comments after. Notice that a coherent way of realising this analytic continuation at the level of the values of the quantum group invariants would give us a way of defining $\mathbb{C}$-valued knot invariants associated with infinite dimensional unitary representations.

However, at least in the case $G = SL(2, \mathbb{R})$ or $G = SL(2, \mathbb{C})$, for some interesting infinite dimensional unitary representations the power series that we obtain have zero radius of convergence. The process of Borel re-summation is analysed in the case of the z-coloured Jones polynomial. It is observed that in this case the power series are of Gevrey type 1, which means their Formal Borel Transform defines an analytic function in a neighbourhood of zero. In particular we can define a re-summation operator apart from exponentially decreasing terms, for example from expression (48). However for the Borel re-summation procedure to be applied in full generality, we need the extra conditions that the formal Borel transform can be analytically continued to some ray $e^{i\theta}\mathbb{R}_0^+$ admitting a Laplace transform with a non zero domain along this direction. This reduces the uncertainty in the re-summation to a numerable set of functions differing by rapidly decreasing terms. It is verified that this is indeed what happens when considering torus knots. Namely the formal Borel transform can be continued along mostly all directions. The singularities of it are ramifying, and the result of re-summation can have at most two branches, both solutions of the problem of analytically extending the values of the coloured Jones polynomial to complex spins. In general for representations in the principal series with real h the two branches define conjugated complex numbers. The question that arises naturally concerns the behaviour of the z-coloured Jones polynomial in the general case, namely:

1. Can the Formal Borel Transform of $\frac{J^z}{2z+1}(K)$ be analytically continued in general?
2. Does it admit Laplace transforms with non zero domain?
3. Which kind of ambiguities will result from this process of re-summation?

4. Do these ambiguities have a geometric meaning[7]? For example which geometric object corresponds to the residue, if any, of the first singularity of the formal Borel Transform of the z-coloured Jones polynomial.

5. Can we generalise all this to links?

These open problems start a totally new direction in quantum group theory with wide applications to mathematical physics.

[7] I must thank Professor Laurent Fredeil for opening my eyes to this

5 Appendix to chapters 1, 2, 3 and 4

5.1 Proof of Theorem 2

This is a known result, first referred to by Kontsevich in his memorable article [Ko]. Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ be its universal enveloping algebra. Let also $\Delta : U(\mathfrak{g}) \to U(\mathfrak{g}) \otimes U(\mathfrak{g})$ be the comultiplication map. It is a coassociative algebra morphism which verifies $\Delta(X) = X \otimes 1 + 1 \otimes X, \forall X \in \mathfrak{g}$. For any integer $n \geq 0$, we define $\Delta^{(n)} : U(\mathfrak{g}) \to U(\mathfrak{g})^{\otimes(n+1)}$ as being the n^{th}-iterate of Δ. In other words: $\Delta^{(0)} = \mathrm{id}$, $\Delta^{(1)} = \Delta$, $\Delta^{(2)} = (\Delta \otimes \mathrm{id}) \circ \Delta$, by coassociativity this equals $(\mathrm{id} \otimes \Delta) \circ \Delta$. In general we have $\Delta^{(n+1)} = (\Delta \otimes \mathrm{id}) \circ \Delta^{(n)}$. All iterates of Δ are algebra morphisms. It is easy to verify that:

$$\Delta^{(n)}(X) = X \otimes 1^{\otimes n-1} + 1 \otimes X \otimes 1^{\otimes n-2} + ... + 1^{\otimes n-1} \otimes X, X \in \mathfrak{g}. \tag{59}$$

The Hopf algebra $U(\mathfrak{g})$ is cocommutative. For a permutation $\sigma \in S_n$, let τ_s : $U(\mathfrak{g})^{\otimes n} \to: U(\mathfrak{g})^{\otimes n}$ be the obvious permutation morphism. From equation (59) it is trivial that $\tau_\sigma \circ \Delta^n(X) = \Delta^n(X)$ for any permutation σ and $X \in \mathfrak{g}$. Therefore:

Lemma 38 *For any permutation $\sigma \in S_n$ we have $\tau_\sigma \circ \Delta^{(n)} = \Delta^{(n)}$.*

This is also true for any $X \in U(\mathfrak{g})$ since $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ as an algebra.

Definition 39 *Let $\mathfrak{g}$ be a Lie algebra. We define $U(\mathfrak{g})^{\otimes n}_{\mathfrak{g}}$ as*

$$U(\mathfrak{g})^{\otimes n}_{\mathfrak{g}} = \{a \in U(\mathfrak{g})^{\otimes n} : [a, \Delta^{(n)}(X)] = 0, \forall X \in U(\mathfrak{g})\}.$$

The tensors in $U(\mathfrak{g})^{\otimes n}_{\mathfrak{g}}$ will be called $\mathfrak{g}$-invariants.

Notice that $U(\mathfrak{g})^{\otimes n}_{\mathfrak{g}}$ is the centre of $U(\mathfrak{g})$ for $n = 1$. Obviously $U(\mathfrak{g})^{\otimes n}_{\mathfrak{g}}$ is a subalgebra of $U(\mathfrak{g})^{\otimes n}$. In addition, since $U(\mathfrak{g})$ is generated by $\mathfrak{g}$, as an algebra we have

$$U(\mathfrak{g})^{\otimes n}_{\mathfrak{g}} = \{a \in U(\mathfrak{g})^{\otimes n} : [a, \Delta^{(n)}(X)] = 0, \forall X \in \mathfrak{g}\}.$$

Define $m : U(\mathfrak{g})^{\otimes n} \to U(\mathfrak{g})$ by $x_1 \otimes x_2 \otimes ... \otimes x_n \mapsto x_1 x_2 ... x_n$.

Lemma 40 *If $a \in U(\mathfrak{g})^{\otimes n}_{\mathfrak{g}}$ then $m(a)$ is central in $U(\mathfrak{g})$*

Proof. Let $X \in \mathfrak{g}$ and $a \in U(\mathfrak{g})^{\otimes n}_{\mathfrak{g}}$. We have $\Delta^{(n)}(X)a = a\Delta^{(n)}(X)$, where $a = a_1 \otimes a_2 \otimes ... \otimes a_n$, since a is $\mathfrak{g}$-invariant; thus $m(\Delta^{(n)}(X)a) = m(a\Delta^{(n)}(X))$. From equation (59), the last equality implies $Xa = aX$, where $X \in \mathfrak{g}$. Therefore, the result follows from the fact $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ as an algebra.

■

Lemma 41 *For a tensor $t = \sum_k a_k \otimes b_k \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ and $i < j \leq n$ let $t_{i,j}$ denote the tensor of $U(\mathfrak{g})^{\otimes n}$ obtained by inserting t in the i^{th} and j^{th} variables and 1 in the others. Suppose $t \in U(\mathfrak{g}) \otimes U(\mathfrak{g})$ is $\mathfrak{g}$-invariant. Then $t_{i,j}$ is also $\mathfrak{g}$-invariant*

Proof. Let σ be a permutation sending 1 to i and 2 to j. For any $X \in U(\mathfrak{g})$ we have, by Lemma 38:

$$[\Delta^{(n)}(X), t_{i,j}] = \tau_\sigma \circ [\Delta^{(n)}(X), t_{1,2}] = \tau_s \circ [(\Delta \otimes \mathrm{id}) \circ \Delta^{(n-1)}(X), t \otimes 1] = 0.$$

■

Therefore

Proposition 42 *Let ω be a chord diagram with n chords and t an infinitesimal R-matrix. Chose an initial point p on the circle to evaluate $\phi_{t,p}(w)$. This element is central in $U(\mathfrak{g})$*

Proof. Trivial from the fact $\phi_t(\omega) = m(\prod_{k=1}^{n} t_{i_k,j_k})$, since an infinitesimal R-matrix is always $\mathfrak{g}$-invariant. Notice that the set of $\mathfrak{g}$-invariant tensors is always an algebra.

■

We now prove that $\phi_{t,p}(\omega)$ does not depend on the initial point p in the circle. Let $4T_p$ be the $4T$ relations now understood modulo isotopies of the circle fixing p. We have already observed in 1.3 that:

Lemma 43 *Let p be an initial point on the circle. The map $\omega \mapsto \phi_{t,p}(\omega)$ satisfies the $4T$ relations.*

This is actually a consequence of equation (1). We now need to prove that moving p around the circle and passing through an endpoint of a chord is a consequence of the $4T_p$-relations. This will also prove that the multiplication of two chord diagrams does not depend on the point in which we break the circles. This argument appears in several places, for example [K] and [BN]. We show that the move in figure 8 is a consequence of the $4T_p$-relations. Call the chord diagram on the left ω and the chord diagram on the right w'. We now understand chord diagrams modulo isotopy fixing p. To this end, let d be a chord of w different of c. Define w_d as in figure 9. In particular $w_d = 0$ modulo the $4T_p$ relations. But obviously

$$\sum_{\substack{d\in\omega \\ d\neq c}} w_d = w - w'$$

modulo isotopy fixing p. This finishes the proof the $\phi_{t,p}(\omega)$ does not depend on p. The rest of the proof of theorem 2 is trivial.

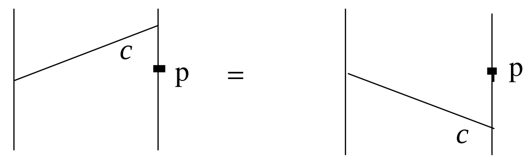

Figure 8: Moving the initial point across an endpoint of a chord

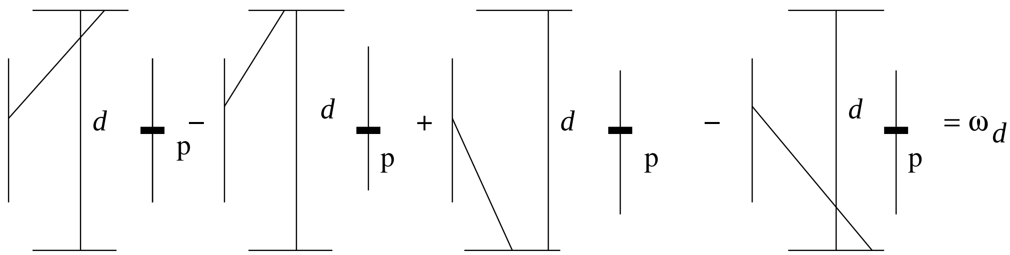

Figure 9: Construction of w_d

5.2 Evaluating the $\mathfrak{sl}(2,\mathbb{C})$ weight system

Our main reference for this section is [CV]. Let ω be a chord diagram and t be an infinitesimal R-matrix in some Lie algebra $\mathfrak{g}$. Recall the evaluation $\phi_t(\omega)$. It satisfies the 4-term relations thus it descends to a morphism from the algebra $\mathcal{A}$ of chord diagrams to the universal enveloping algebra of $\mathfrak{g}$. Depending on the Lie algebra $\mathfrak{g}$, and the infinitesimal R-matrix t, this evaluations may satisfy some additional relations. Let us consider the case $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ and s is the infinitesimal R-matrix coming from a quarter of Cartan-Killing form, thus

$$s = E \otimes F + F \otimes E + \frac{H \otimes H}{2}.$$

Therefore, $s = -4t$ where t is the infinitesimal R-matrix defined in 1.4.6. The use of this normalisation will make the formulae simpler. Notice $\phi_t(w) = (1/(-4)^n)\phi_s(w)$

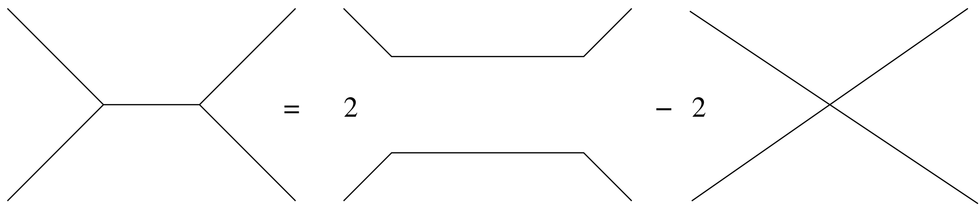

Figure 10: 3-Term Relations for the $\mathfrak{sl}(2,\mathbb{C})$-weight system

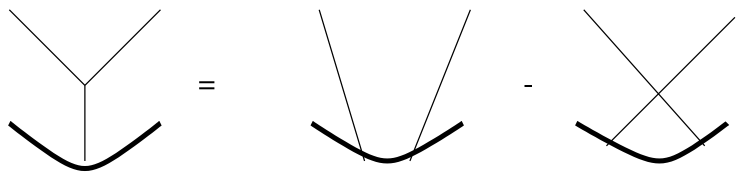

Figure 11: STU-relation

if ω is a chord diagram with n-chords, thus we can easily go back to our old normalisation. The evaluation of ϕ_s satisfies a set of 6-term relations, consequence of a 3-term relation for the $\mathfrak{sl}(2,\mathbb{C})$-weight system, and the general STU-relation. These relations are shown in figures 10 and 11. The $\mathfrak{sl}(2,\mathbb{C})$-relations are here shown at the level of Chinese Character Diagrams. See [BN] or [CV]. We refer also to [CV] for the complete set of 6-term relation, consequence of all the ways we can solve the vertices in the 3-term relations using the STU relation. A major consequence of these relations is a recursion formula for the evaluation of the $\mathfrak{sl}(2,\mathbb{C})$-weight system, which we explain now: Let ω be a chord diagram. It consists of a set of pairs of distinct points in the circle. Any pair of points defines a chord, the segment connecting them. Let a be a chord of ω, it divides the circle into two half circles. Let $\omega(a)$ be the chord diagram obtained from ω by removing the chord a. Let $a_1, ..., a_n$ be the chords of ω intersecting a. They have endpoints e_i and f_i where all e_i's (resp

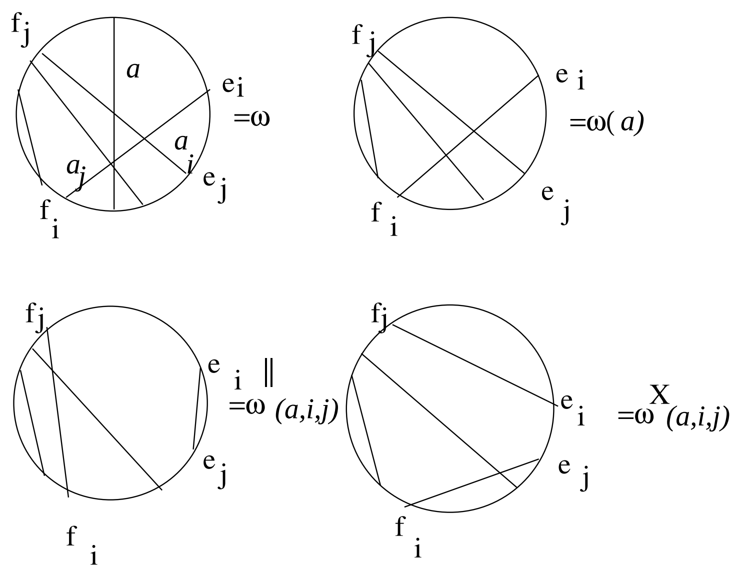

Figure 12: Diagrams appearing in the recursive relation for the $\mathfrak{sl}(2,\mathbb{C})$-weight system

f_i's) belong to the same half circle. Let a_i and a_j be two distinct chords intersecting a. We define $w^{||}(a,i,j)$ (resp $w^X(a,i,j)$) as being obtained from $w(a)$ by removing a and a_i, and inserting two chords connecting e_i to e_j and f_i to f_j (resp e_i to f_j and e_j to f_i). See figure 12. Recall $\ominus$ stands for the chord diagram with only one chord. Theorem 1 of [CV] tells us:

Theorem 44 (Recursive Relation for $\mathfrak{sl}(2,\mathbb{C})$-weight system) *Let w be a chord diagram and a be a chord of it. Let also $a_1,..,a_n$ be the chords of ω intersecting a. We have:*

$$\phi_s(\omega) = \phi_s\left((\ominus - 2n)\omega(a) + 2\sum_{1\leq i<j\leq n}(\omega^{||}(a,i,j) - \omega^X(a,i,j))\right)$$

5.3 Proof of Theorem 26

The main results for the proof of this theorem appear in [L] VI.§5.

Consider the following elements of $\mathfrak{sl}(2,\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C})$:

$$W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad E^- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad E^+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \tag{60}$$

Thus they are a basis of $\mathfrak{sl}(2,\mathbb{C})$ and satisfy the commutation relations:

$$[E^+, E^-] = -4iW, \quad [W, E^+] = 2iE^+, \quad [W, E^-] = -2iE^-. \tag{61}$$

Consider

$$\mathcal{H}_0 = \bigoplus_{n\in\mathbb{Z}} \mathrm{span}\{u_{2n}\}, \quad \mathcal{H}_1 = \bigoplus_{n\in\mathbb{Z}} \mathrm{span}\{u_{2n+1}\}.$$

Let $s \in \mathbb{C}$, consider the representations $r(s,0)$ and $r(s,1)$ of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathcal{H}_0$ and $\mathcal{H}_1$, respectively, such that:

$$Wu_n = inu_n, \quad E^- u_n = (s+1-n)u_{n-2}, \quad E^+ u_n = (s+1+n)u_{n+2}. \tag{62}$$

Lemma 45 *For any $s \in \mathbb{C}$ and $\varepsilon \in \{0,1\}$, the representation $r(\rho,\varepsilon)$ have a central character $c(s,\varepsilon)$. In other words, any element X of the centre $\mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C}))) \to \mathbb{C}$ acts in then as a multiple $c(s,\varepsilon)(X)$ of the identity.*

Notice the representations $r(s,\varepsilon)$ are not always irreducible.

Proof. Suppose first that $s \in \mathbb{C} \setminus \mathbb{Z}$. Let $f : \mathcal{H}_\varepsilon \to \mathcal{H}_e$ be an intertwiner, we show that it needs to be a multiple of id. This will prove that $r(s,\varepsilon)$ has a central character. Let $f(u_n) = \sum_k A_n^k u_k$. Since $f \circ W = W \circ f$ we have:

$$in \sum_k A_n^k u_k = \sum_k ikA_n^k u_k, \forall n \implies i(n-k)A_n^k = 0, \forall n, k.$$

In particular $A_n^k = 0$ if $k \neq n$. Let us now show that $A_n^n = A_{n+2}^{n+2}, \forall n$. Given that $f \circ E^+ = E^+ \circ f$, yields $(s+1+n)A_n^n = (s+1+n)A_{n+2}^{n+2}$, thus, since $s \notin \mathbb{Z}$, we must have $A_n^n = A_{n+2}^{n+2}, \forall n$. For the general case, consider the matrix elements:

$$M^{s,\varepsilon}(X,k,n) = \langle u^n | r(s,\varepsilon)(X) | u_k \rangle ,$$

where $X \in U(\mathfrak{sl}(2,\mathbb{C}))$. Then obviously $M^{s,\varepsilon}(X,k,n)$ is a polynomial in s if k,n,ε and X are fixed. If $X \in \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C})))$ then $M^{s,\varepsilon}(X,k,n) = \delta(k,n)c(s,\varepsilon)(X)$ for $s \in \mathbb{C} \setminus \mathbb{Z}$, where $c(s,\varepsilon)(X)$ is a polynomial in s, uniquely determined. In particular we must also have $M^{s,\varepsilon}(X,k,n) = \delta(k,n)c(s,\varepsilon)(X)$ for $s \in \mathbb{Z}$ and $X \in \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C})))$. In other words

$$r(s,\varepsilon)(X) = c(s,\varepsilon)(X)\mathrm{id}, \forall X \in \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C}))), \forall s \in \mathbb{C}, \forall \varepsilon \in \{0,1\},$$

where $c(s,\varepsilon)(X)$ is a polynomial in s. ■

Suppose $s = m$, where m is a positive odd integer. Then the representation $r(m,0)$ has two invariant subspaces, namely

$$\mathcal{H}_0(\leq -(m+1)) = \bigoplus_{\substack{n \in \leq -(m+1) \\ n \text{ is even}}} \mathrm{span}\{u_n\} \text{ and } \mathcal{H}_0(\geq (m+1)) = \bigoplus_{\substack{n \in \geq (m+1) \\ n \text{ is even}}} \mathrm{span}\{u_n\}.$$

We thus have representations $r(m,0,\leq)$ and $r(m,0,\geq)$ of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathcal{H}_0(\leq -(m+1))$ and $\mathcal{H}_0(\geq (m+1))$ for any odd m. Let

$$\mathcal{H}_0(m)_{\mathrm{fin}} = \frac{\mathcal{H}_0}{\mathcal{H}_0(\leq -(m+1)) \oplus \mathcal{H}_0(\geq -(m+1))},$$

then $r(m,0)$ descends to a finite dimensional representation $r(m,0)_{\mathrm{fin}}$ of $\mathfrak{sl}(2,\mathbb{C})$ in it. As in lemma 45 one proves it is irreducible, for any intertwiner must be a multiple of id. Therefore $r(m,1)_{\mathrm{fin}}$ is the representation of $\mathfrak{sl}(2,\mathbb{C})$ of spin $(m-1)/2$

since $\mathcal{H}_0(m)_{\text{fin}}$ has dimension m. Obviously, all representations $r(m,0)_{\text{fin}}$, $r(m,0,\leq)$ and $r(m,0,\geq)$ have central characters and they are all equal to $c(m,0)$. This is a consequence of Lemma 45.

Let λ_z be the central character of the representation of $\mathfrak{sl}(2,\mathbb{C})$ of highest weight $2z$, in other words of the representation of $\mathfrak{sl}(2,\mathbb{C})$ of spin z, where z is a complex number. The last observations prove immediately that $c(m,0) = \lambda_{(m-1)/2}$ if m is a positive odd integer, since both equal the central character of the representation of $\mathfrak{sl}(2,\mathbb{C})$ in $\mathcal{H}_0(m)_{\text{fin}}$. Let $X \in \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C})))$, thus both $c(s,0)(X)$ and $\lambda_{(s-1)/2}(X)$ are polynomials in s. Since they coincide for $m \in \{1,3,5,..\}$ we have proved

Lemma 46 $c(z,0) = \lambda_{(z-1)/2}, \forall z \in \mathbb{C}$

We prove analogously:

Lemma 47 $c(z,-1) = \lambda_{(z-1)/2}, \forall z \in \mathbb{C}$

The following theorem is shown in [L].

Theorem 48 *Let $s \in i\mathbb{R}$ and $\varepsilon \in \{0,1\}$. The representations $r(s,e)$ is a subrepresentation of $R^{\infty s,\varepsilon}$, the representation of $\mathfrak{sl}(2,\mathbb{C})$ in the space of smooth vectors of $R^{s,\varepsilon}$. If m is an odd negative integer, then the representation $r(m,0,\geq)$ (resp $r(m,0,leq)$ is a subrepresentation of $R^{\infty m,+}$ (resp $R^{\infty,m,-}$), and analogously for $r(-m,-1,\geq)$ and $r(-m,-1,\leq)$ if m is an even negative integer.*

Theorem 26 now follows easily.

5.4 Entire functions of exponential order and proof of Theorem 34 for the Figure of Eight Knot

5.4.1 Power series developments of functions of exponential order

All the material presented here can be found in mostly all books of advanced complex analysis. My favourite one is [CH].

Definition 49 *Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function, in other words a function analytic in $\mathbb{C}$. Then f is said to be of exponential order if there exist $B, C < \infty$ such that*

$$|f(z)| \leq B \exp(C|z|), \forall z \in \mathbb{C}.$$

Definition 50 *Let f be an entire function of exponential order. The function f is said to be of type $C > 0$ if we can find $B < \infty$ such that*

$$|f(z)| \leq B \exp(C|z|), \forall z \in \mathbb{C}.$$

Therefore, if f and g are two functions of exponential order of type A and B, then also $f+g$ and fg are of exponential order of types $\max(A, B)$ and $A+B$, respectively. Obvioulsly the functions $\sinh(Az), \cosh(Az)$ or $\exp(Az)$ are all of exponential order and of type A. If n is integer and $A \in \mathbb{C}$, then

$$\frac{\sinh((n+1)Ax)}{\sinh(Ax)} = \sum_{k=-n}^{n} \exp(Akx),$$

which proves that $F_{m,p,z}(x)$ (see 4.1) is an entire function of exponential order if $2z + 1$ is a non zero integer.

An easy application of Cauchy's integral formula is the following known fact about entire functions: Let f be an entire function. Put

$$f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{63}$$

Lemma 51 (Cauchy's Inequality) *Let f be an analytic function in a neighbourhood of the origin in $\mathbb{C}$. If $f(z) \le M$ in the circle of radius R around the origin. We have*

$$|a_n| \le \frac{M}{R^n}, \forall n \in \mathbb{N}.$$

Therefore

Theorem 52 *Suppose f is an entire function such that $|f(z)| \le A\exp(Cz)$, for some $A, C > 0$. Then the coefficients of (63) satisfy*

$$a_n \le AC^n \frac{e^n}{n^n}, \forall n \in \mathbb{N},$$

thus from Lemma 72 (consequence of Stirling Inequalities 93), we can find $D < \infty$ such that

$$|a_n| \le \frac{D^n}{n!}, \forall n \in \mathbb{N}.$$

This constant D can be taken as close to C as we want. This is obvious from Stirling Inequalities.

Proof. See [CH] page 254. By Cauchy's Inequalities we have

$$|a_n| \le A\frac{\exp(CR)}{R^n}, \forall n \in \mathbb{N},$$

for any $R > 0$. Thus

$$|a_n|^{1/n} \le A^{1/n}\exp(CR/n - \ln(R)), \forall R > 0, \forall n \in \mathbb{N}.$$

The minumum, which obviously exists since $C > 0$, of $CR/n - \ln(R)$ when $R > 0$ is attained in $R = n/C$. In particular

$$|a_n|^{1/n} \leq A^{1/n}\frac{1}{n}eC, \forall n \in \mathbb{N},$$

which implies

$$|a_n| \leq AC^n\frac{e^n}{n^n}, \forall n \in \mathbb{N}.$$

■

5.4.2 Proof of the Theorem 34 For the Figure of Eight Knot

Recall that the z-Coloured Jones Polynomial for the zero framed Figure of Eight Knot is

$$\frac{J^z}{2z+1}(4_1) = \frac{1}{2z+1}\frac{q^{\frac{2z+1}{2}} - q^{-\frac{2z+1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}\sum_{n=0}^{+\infty} D(n,z).$$

where

$$D(n,z) = \prod_{k=1}^{n}\left[\left(q^{\frac{2z+1}{2}} - q^{-\frac{2z+1}{2}}\right)^2 - \left(q^{\frac{k}{2}} - q^{-\frac{k}{2}}\right)^2\right].$$

and $q = \exp(h)$. See 3.3.1. Let us prove it is a power series of Gevrey type 1. To make the notation simpler, we only analyse the case $2z + 1 = 0$. The general case is totally similar. Notice that at $z = 0$ we attain an infinite dimensional representation of $SL(2,\mathbb{R})$, see Theorem 26. Since the set of power series of Gevrey type 1 is a subalgebra of $\mathbb{C}[[h]]$ it suffices to prove that

$$\sum_{N=0}^{\infty} c_N h^N = \sum_{n=0}^{\infty}(-1)^n 2^{2n}\prod_{m=1}^{n}\sinh^2\left(\frac{mh}{2}\right) \tag{64}$$

is a power series of Gevrey type 1. Notice this power series only has even terms. Let

$$F_N(h) = \sum_{n=0}^{N/2} 2^{2n}(-1)^n \prod_{m=1}^{n} \sinh^2\left(\frac{mh}{2}\right),$$

thus obviously $C_N = a_N$ where a_N is the N^{th} order term of the Taylor expansion (63) of $F_N(h)$ at $h = 0$. We now wish to apply Theorem 52 to F_N. For any $h \in \mathbb{C}$ we have $|\sinh(Ah)| \leq \exp(A|h|), h \in \mathbb{C}$. Thus for any $h \in \mathbb{C}$ we have:

$$\begin{aligned} |F_N(h)| &\leq \sum_{n=0}^{N/2} 2^{2n} \prod_{m=1}^{n} \exp(m|h|) \\ &= \sum_{n=0}^{N/2} 2^{2n} \exp\left(\frac{n(n+1)}{2}|h|\right) \\ &\leq \left(\frac{N}{2}+1\right) 2^N \exp\left(\frac{N(N+2)}{8}|h|\right). \end{aligned}$$

In particular, by theorem 52 we have

$$|a_N| \leq \left(\frac{N}{2}+1\right) 2^N \left(\frac{N(N+2)}{8}\right)^N \frac{e^N}{N^N} \leq C^N N^N,$$

where C does not depend on N. Therefore by Lemma 72 there exists a $D < \infty$ such that

$$|C_N| = |a_N| \leq D^N N!, \forall N \in \mathbb{N}_0$$

this proves Theorem 34 for the Figure of Eight knot.

The same kind of argument would yield a similar result for the Trefoil knot, see 3.3.1. Even though equation (37) and Theorem 29 have already given a proof of it.

In fact we can prove:

Theorem 53 *Let K be a framed knot. Consider Habiro's general expression (26) for the z-Coloured Jones Polynomial of it as in 3.3.1. Suppose that the exists A*

and B such that for any integer n the function $f_n(K)(z)$ is an entire function of exponential order for which:

$$|f_n(K)(z)| \leq A\exp(Bn^2|z|), \forall z \in \mathbb{C}, \forall n \in \mathbb{N}_0.$$

Then $\frac{J^z}{2z+1}(K)(h)$ is a power series of Gevrey type 1.

Therefore with a better knowledge of the coefficients of Habiro's formula (26) we may prove Theorem 34 in full generality and in a simple way. This would avoid the complicated and especially very techical approach for it in chapter 6.

5.5 Condensed proof of Theorem 34

This is extracted from [FM1]. Chapter 6 is an expanded version of this section with full proofs.

We assume that the reader is familiar with the construction of the Kontsevich Integral, as well as the algebraic structure in the space of chord diagrams. See for example [BN], [Wi1] or [C]. Let $\mathcal{A}_{\text{fin}} = \oplus_{n\in\mathbb{N}_0}\mathcal{A}_n$. The connected sum of chord diagrams defines a product $\mathcal{A}_n \otimes \mathcal{A}_m \to \mathcal{A}_{m+n}$, providing $\mathcal{A}_{\text{fin}}$ a graded algebra structure. By definition, the algebra of chord diagrams $\mathcal{A}$ is the graded completion of $\mathcal{A}_{\text{fin}}$, or alternatively the algebra of formal linear combinations of the form $\sum_n w_n h^n$ where $w_n \in \mathcal{A}_n, \forall n \in \mathbb{N}_0$.

The unframed Kontsevich Integral has values on the algebra $\mathcal{A}'$, defined similarly to the algebra $\mathcal{A}$ of chord diagrams, but considering also the framing independence condition of figure 13. Alternatively we can define $\mathcal{A}'$ as the graded completion of $\mathcal{A}'_{\text{fin}} = \mathcal{A}_{\text{fin}}/ < \ominus\mathcal{A}_{\text{fin}} >$, the last with the obvious grading, see [Wi2]. Here $\ominus$ denotes the chord diagram with just one chord.

Consider a knot K made out of n tangle generators G, like the ones shown in figure 14, on top of each others, with a chosen orientation. They can be of six different kinds, namely $\cap$, $\cup$, C_-, C_+, X_- and X_+. We can suppose they have height 1. The knot K is a Morse knot, and the critical points of it are contained in $\{0, ..., n\}$. Let $I_k = [k-1, k]$, where $k = 1, ..., n$.

Define:

$$Z(K) = \sum_{m=0}^{\infty} \frac{h^m}{(2\pi i)^m} \int_{\substack{0<t_1<t_2<...<t_m<n \\ t_j \notin \{0,...,n\}, j=1,...,m}} \sum_{\substack{\text{pairings} \quad P \\ P=\{\{z_j,z_j'\}:I_{k_j} \to \mathbb{C}\}_{j=1}^m}} (-1)^{\#\downarrow P} w_P \bigwedge_{j=1}^{m} \frac{dz_j - dz_j'}{z_j - z_j'} \in \mathcal{A}', \quad (65)$$

that is

$$Z(K) = \sum_{m=0}^{\infty} h^m \sum_{\substack{\text{pairings} \quad P \\ P=\{\{z_j,z_j'\}:I_{k_j} \to \mathbb{C}\}_{j=1}^m}} Z(P, K) w_P \in \mathcal{A}'. \quad (66)$$

Here I_{k_j} denotes the interval where the pair of functions $p_j = \{z_j, z_j'\} : I_{k_j} \to \mathbb{C}$ (a chord) is defined, thus $k_1 < k_2 < ... < k_m$. If P is a pairing then w_P denotes the chord diagram constructed out of it as in figure 15, whereas $\downarrow P$ denotes the set of z_i or z_i' where the orientation of K points downwards. The framing independence relation ensures the integrals are convergent, for the divergent integrals evaluate to zero in $\mathcal{A}'$, see [BN] 4.3.1.

Any pairing P with m chords is defined in some connected component of $\{0 < t_1 < t_2 < ... < t_m < n, t_j \notin \{0, ..., n\}, j = 1, ..., m\}$, in particular there are m_k chords $p_i = \{z_i, z_i'\}$ defined in each I_k for $k \in \{1, ..., n\}$. We can thus index the chords as $p_i^k, k = 1, ...n, i = 1, ...m_k$. Let $\underline{m} = (m_1, ..., m_k)$, thus $|\underline{m}| = m_1 + ... + m_n = m$. We say a chord p defined in a I_k associated with a generator G_k of the kind $\cup$ or $\cap$ is

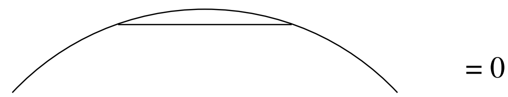

Figure 13: Framing independence relation on chord diagrams

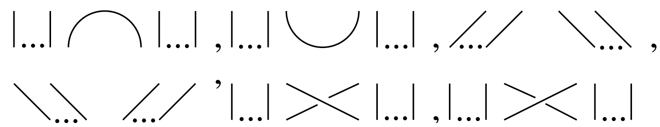

Figure 14: Generator tangles of the kinds $\cap$, $\cup$, C_-, C_+, X_- and X_+

of type $T(p) = A$ (resp $T(p) = B$) if it looks like the one in figure 16 (resp the ones in figure 17). If the generator G_k is of type C_-, C_+, X_- or X_+ then by definition all the chords in I_k are of type B. If G_k is a generator of type $\cup$, X_+ or X_-, let $T^k = \big(T(p_1^k), ..., T(p_{m_k}^k)\big)$. If G_k is of type $\cap$ then we define $T^k = \big(T(p_{m_k}^k), ..., T(p_1^k)\big)$. Let $B(k)$ be the number of chords in I_k of type B and $B(P)$ be the total number of B-chords in P. Notice that since we are considering the framing independence relation we can suppose $T_1^k = B, k = 1, ..., n$. This is a necessary condition for all the integrals to be convergent, in the first place.

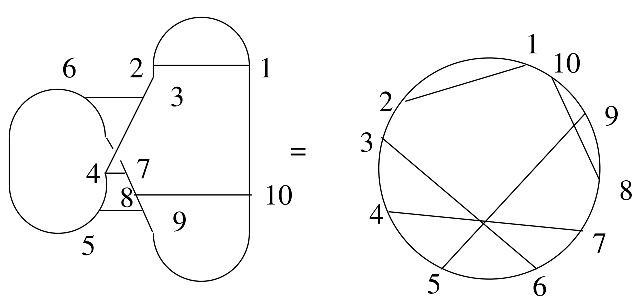

Figure 15: Defining a chord diagram out of a pairing.

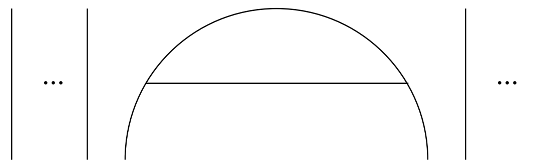

Figure 16: A chord of type A

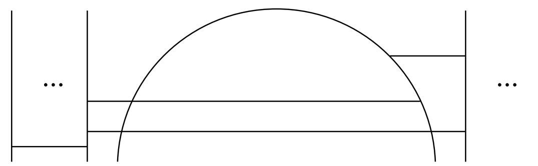

Figure 17: Chords of type B

Explicit calculations prove:

Lemma 54 [8] *Fix a knot K as before, there exists a $C < +\infty$ such that for any pairing P with m chords we have:*

$$|Z(P,K)| \leq C^m \prod_{k=1}^{n} \frac{1}{B(k)!} \prod_{k=1}^{n} \prod_{\substack{i \in \{1,...,m_k\} \\ T_i^k = A}} \frac{1}{\#\{j \in \{1,..,i\} : T_j^k = B\}}. \tag{67}$$

Proof. (sketch) Let $T = (T_1, .., T_n)$ be a sequence of A's and B's with $T_1 = B$. Let $B(T)$ be the number of elements of T equal to B, and let $C(A) = 1$ and $C(B) = 1/2$. We have:

$$I(T) = \int_{0<t_1<...<t_m<1} \prod_{i=1}^{m} \frac{1}{t_i^{C(T(A_i))}} dt_1 ... dt_m = 2^m \frac{1}{B(T)!} \prod_{\substack{i \in \{1,...,m\} \\ T_i = A}} \frac{1}{\#\{j \in \{1,...,i\} : T_j = B\}}. \tag{68}$$

[8] See Theorem 69.

One proves this from the equality

$$\int_{-\infty<x_1<x_2<...<x_n<0} e^{\sum_{i=1}^n \lambda_i x_i} dx_1...dx_n = \frac{1}{\lambda_1(\lambda_1+\lambda_2)...(\lambda_1+\lambda_2+...+\lambda_n)}, \qquad (69)$$

(easy to prove by induction if $\lambda_1, \lambda_2, ..., \lambda_n > 0$) by a continuity argument. Consider the generators $\cup$ and $\cap$ to be made of semicircles of radius 1 and strings parallel to z-axis. Unpacking equation (66) yields a product of n integrals, one for each I_k, equal to, or bounded by, integrals like $C^{m_k} I(T)$, where C is fixed. ■

Let $z \in \mathbb{C}$ be a complex number, and let λ_z be the central character of the representation of $\mathfrak{sl}(2,\mathbb{C})$ of highest weight $2z$. We wish to give an estimate for $|\lambda_z(\phi_s(w_P))|$, where s denotes the infinitesimal R-matrix in $\mathfrak{sl}(2,\mathbb{C})$ coming from a quarter of the Cartan-Killing form and w_P is the chord diagrams made out of the pairing P. Even though $\phi_s(w)$ does not make sense in $\mathcal{A}'$, since ϕ_s does not satisfy the framing independence relations, w_P does define a chord diagram, thus an element of $\mathcal{A}$, and in particular it makes sense to consider $\phi_s(w_P)$.

Recall the recursive evaluation of the $\mathfrak{sl}(2,\mathbb{C})$ weight system $\phi_s : \mathcal{A} \to U(\mathfrak{sl}(2,\mathbb{C}))$ in [CV], theorem 1 [9]. Let w be a chord diagram and let a be a chord of it, thus a divides the circle supporting w into two semicircles. Define $C(a,w)$ as the set of chords of w that cross a. It has cardinality $x(a,w)$. We define $w(a)$ as the chord diagram obtained from w by removing a. Let b and c be distinct chords in $C(a,w)$, they have endpoints e_a, e_b, f_a, f_b such that e_a and e_b (resp f_a and f_b) lie in the same semicircle. Define $w^X(a,b,c)$ (resp $w^{||}(a,b,c)$) as the chord diagrams obtained from w through removing a,b and c and adding two new chords joining e_a with f_b and e_b

[9] See Theorem 44.

with f_a (resp e_a with e_b and f_a with f_b). Theorem 1 of [CV] tells us:

$$(\lambda_z \circ \phi_s)(w) = (\lambda_z \circ \phi_s)(\ominus w(a) - 2x(a,w)(\lambda_z \circ \phi_s)(w(a)) + \sum_{\substack{b,c \in C(a,w) \\ b \neq c}} 2\left((\lambda_z \circ \phi_s)(w^X(a,b,c)) - (\lambda_z \circ \phi_s)(w^{||}(a,b,c))\right). \quad (70)$$

Let w be a chord diagram with m chords having a set $\alpha(w)$ of chords that do not cross each others (we do not suppose that this set is maximal). Pick up a chord $a_0 \in \alpha(w)$. Consider a diagram w' appearing in the recursive evaluation (70). Then $\alpha(w') = \alpha(w) \setminus \{a_0\}$ is a set of chords of w' that do not intersect each others. In addition we always have $x(a,w') \leq x(a,w), \forall a \in \alpha(w')$. We need to use the fact that the chords of $\alpha(w)$ do not cross each others to prove this. An obvious induction based on equation (70) and this last fact tells you:

$$|(\lambda_z \circ \phi_s)|(w) \leq S_z(m - \#\alpha(w)) \prod_{a \in a(w)} \left(|c_z| + 2x(a,w)^2 + x(a,w)\right) \quad (71)$$

$$\leq S_z(m - \#\alpha(w)) C_z^{\#\alpha(w)} \prod_{a \in a(w)} (1 + x(a,w))^2 . \quad (72)$$

Here $c_z = (\lambda_z \circ \phi_s)(\ominus)$ and $C_z = \max(|c_z|, 2)$. In general, if $i \in \mathbb{N}$, we define $S_z(i) = \max\{|\lambda_z \circ \phi_s(x)|, x \in W_i\}$ where W_i is the set of chord diagrams with k chords. The recursive evaluation tells you again that $S_z(i) \leq \prod_{j=1}^{i}(|c_z| + 2j^2 + j) \leq C_z^j (j+1)!^2$. Given this last estimates we prove:

Lemma 55 *Let $z \in \mathbb{C}$ there exist a constant $C_z < +\infty$ such that the following is true: Let w be a chord diagrams with m chords having a set $\alpha(w)$ of chords that do not cross each others. We have the estimate:*

$$|(\lambda_z \circ \phi_s)|(w) < C_z^m (k+1)!^2 \prod_{a \in a(w)} (1 + x(a,w))^2 , \quad (73)$$

where $k = m - \#\alpha(w)$. If in addition $x(a, w) \neq 0, \forall a \in \alpha(w)$ we can write this in a more useful form for later, namely

$$|(\lambda_z \circ \phi_s)|(w) < (2C_z^m)(k+1)!^2 \prod_{a \in a(w)} (x(a,w))^2 . \tag{74}$$

Let K be an oriented knot and let P be a pairing with m chords. We want to apply the lemma above to w_P, the chord diagram made out of P. Each chord p of P gives rise to a chord a_p of w_P. Two chords $a_p, a_{p'}$ with p and p' chords of type A cannot cross each others, therefore we define $\alpha(w_P) = \{a_p : T(p) = A\}$. If p is defined in a generator G of K of type $\cup$ (resp. $\cap$), then $x(a_p, w_P)$ is the number of B-chords defined in the same generator G staying bellow (resp. above) p, thus $x(a_p, w_P)$ can be supposed to be different of zero by the framing independence relation, since $T_1^k = B$. Combining lemmas 54 and 55, we conclude there exists a $C < +\infty$ such that:

$$|(\lambda_z \circ \phi_s) Z(P, K) w_P| \leq C^m \frac{(B(P)!)^2}{\prod_{k=1}^n B(k)!} \prod_{k=1}^n \prod_{\substack{i \in \{1,\ldots,m_k\} \\ T_i^k = A}} \#\{j \in \{1,..,i\} : T_j^k = B\} \tag{75}$$

where C only depends on z and K. Notice $B(P) = B(1) + \ldots + B(n)$ is the number of chords in P of type B.

An easy consequence of Stirling inequalities is the fact that given $n \in \mathbb{N}$, there exists a $C < \infty$ such that $m! \leq C^m m_1! \ldots m_n!$, where $m_1 + \ldots + m_n = m$, for any $m_1, \ldots, m_n \in \mathbb{N}$. Another consequence is the fact that there exists a $C < +\infty$ such that $m^m < C^m m!, \forall m \in \mathbb{N}$. Let K be a knot and P a pairing with m chords.

Putting everything together we prove

$$
\begin{aligned}
|(\lambda_z \circ \phi_s)(Z(P,K)w_P)| &\leq C^m B(P)^{B(P)} B(P)^{A(P)} && (76)\\
&\leq C^m (A(P)+B(P))^{A(P)+B(P)} && (77)\\
&\leq D^m m!, && (78)
\end{aligned}
$$

for any pairing with m chords. Here $A(P)$ is the number of chords of P of type A. Notice $D < +\infty$ only depends on z and K. We know $s = -t/4$ thus $f_t(w) = (-4)^m \phi_s(w)$, where m is the number of chords of w. therefore we can also find a constant $C < +\infty$ such that:

$$
|(\lambda_z \circ \phi_t)(Z(P,K)w_P)| \leq C^m m!. \tag{79}
$$

We have now done the most difficult part of the proof of Theorem 34, even though this last inequality is not quite enough yet. Let us go back to the definition of the Kontsevich Integral. Let ∞ be a Morse knot equivalent to the unknot but with 4 critical points. Let N be the number of critical points of K. The unframed Kontsevich integral of K is

$$
\mathcal{Z}_u(K) = \frac{Z(K)}{Z(\infty)^{N/2}}. \tag{80}
$$

Let us define the framed version of the Kontsevich Integral. Recall we have a coproduct in $\mathcal{A}_{\text{fin}}$ defined by $D(w) = \sum_{x \subset w} x \otimes (w \setminus x)$, whenever w is a chord diagram. Here x is a chord diagram made out some chords of w and $w \setminus x$ is the complementary diagram. Recall $\mathcal{A}_{\text{fin}}$ has a grading deg where the grading coefficient of a chord diagram is its number of chords. Consider the map $\psi : \mathcal{A}_{\text{fin}} \to \mathcal{A}_{\text{fin}}$ such that

$$
\psi(w) = \sum_{x \subset w} (-\ominus)^{\deg(x)} (w \setminus x). \tag{81}
$$

for a chord diagram w. Thus ψ satisfies the $4T$-relation. In fact ψ is a Hopf algebra projection, see [Wi2] (this is not a trivial fact). The morphism ψ is zero on the ideal generated by $\ominus$, thus ψ defines an algebra morphism $\mathcal{A}'_{\text{fin}} \to \mathcal{A}_{\text{fin}}$. It extends to a morphism $\psi_0 : \mathcal{A}' \to \mathcal{A}$ of the graded completions.

Definition 56 *The Kontsevich Integral of K is*

$$\mathcal{Z}(K) = e^{\ominus F(K)h}\psi_0(\mathcal{Z}_u(K)) \in \mathcal{A}, \tag{82}$$

where $F(K)$ is the framing coefficient of K.

This definition is equivalent to the standard one. See [Wi1] or [LM2] theorem 5.13. Notice $\psi(Z(\infty))^{-1} = \mathcal{Z}(O)$, which is the (framed) Kontsevich Integral of the zero framed unknot. Let $z \in \mathbb{C}$. Since $(\lambda_z \circ \phi_t)$ is an algebra morphism, we have:

$$\begin{aligned}
\frac{J^z(K)}{2z+1} &= (\lambda_z \circ \phi_t)\mathcal{Z}(K) && (83)\\
&= (\lambda_z \circ \phi_s)\left(e^{\ominus F(K)h}\right)(\lambda_z \circ \phi_s)\left(\psi_0(Z(\infty))^{-N/2}\right)(\lambda_z \circ \phi_t)(\psi_0(Z(K))) && (84)\\
&= e^{c_z h}(\lambda_z \circ \phi_t)\left(\mathcal{Z}_f(O)\right)^{N/2}(\lambda_z \circ \phi_t)(\psi_0(Z(K)) && (85)\\
&= e^{c_z h}\left(\frac{1}{2z+1}\frac{\sinh((2z+1)h/2)}{\sinh(h/2)}\right)^{N/2}(\lambda_z \circ \phi_t)(\psi_0(Z(K)). && (86)
\end{aligned}$$

Since the set $G_1[[h]]$ of power series of Gevrey type 1 forms an subalgebra of $\mathbb{C}[[h]]$, the proof of theorem 34 will be finished if we prove that

$$(\lambda_z \circ \phi_t)(\psi_0(Z(K)) = \sum_{m=0}^{\infty} h^m \sum_{\substack{\text{pairings} \quad P \\ P=\{\{z_j, z'_j : I_{k_j} \to \mathbb{C}\}, j=1,\dots,m\}}} (\lambda_z \circ \phi_t)(\psi_0(Z(P,K)w_P) \tag{87}$$

is of Gevrey type 1. The estimate for $|\lambda_z(\phi_s)(w)|$ in 73 continues to hold if we remove a chord from w, maintaining the right hand side of 73 fixed. Immediately we have:

$$|(\lambda_z \circ \phi_s)|(\psi_0(w)) \leq \sum_{x \subset w} |(\lambda_z \circ \phi_s)(-\ominus)|^{\deg(x)} |(\lambda_z \circ \phi_s)(w \setminus x)| \tag{88}$$

$$\leq (2C_z)^m C_z^m (k+1)!^2 \prod_{a \in a(w)} (1 + x(a,w))^2 . \tag{89}$$

Recall that $c_z = (\lambda_z \circ \phi_s)(\ominus)$ and $C_z = \max(|c_z|, 1)$. Notice there are 2^m splittings $w = x \cup (w \setminus x)$ if w is a chord diagram with m chords. As before we prove.

Lemma 57 *The estimate 79 continues to hold if we put $(\psi_0(Z(P,K)w_P)$ instead of $(Z(P,K)w_P)$.*

We now prove $(\lambda_z \circ \phi_t)(\psi_0(Z(K)) \in G_1[[h]]$. Since we have proved Lemma 57, a glance at equation 87 reduces the proof of this to estimating the number of pairings P with m-chords, which is a simple exercise of counting. Let $M = \max_{t \in \mathbb{R}} \#((t \times \mathbb{C}) \cap K)$. Consider pairings P with m chords, having m_i chords in each interval I_i for $i = 1, ..., n$, thus $m_1 + ... m_n = m$. There are at most $(M(M-1)/2)^m$ pairings like this. Recall the classical combinatorics problem which asks about the number of ways we can put X indistinguishable objects into N distinguishable boxes. Its solution is $\frac{(N+X-1)!}{(N-1)!X!}$. In our case we have exactly $X = m$ objects (chords) and $N = n$ boxes (intervals I_k). Thus there are at most $(M(M-1)/2)^m \frac{(n+m)!}{(n-1)!m!}$ pairings P with m chords. Given that n and M are constant, this last term grows exponentially with respect to m. This finishes the proof of theorem 34.

6 On the Kontsevich Integral

The main aim of this Chapter is to describe the Kontsevich Integral in a way such that we can prove Theorem 34

6.1 Definition of the Kontsevich Universal Knot Invariant

We freely use the notation of 1.1. We will work the C^1 category, which permits us to use the Kontsevich integral as the definition of the unframed Kontsevich Universal knot invariant. A (oriented) knot K is an oriented C^1 submanifold of $\mathbb{R}^3$ diffeomorphic with the oriented circle S^1. Two knots K and K' are said to be isotopic if there exists a C^1 diffeomorphism $f:\mathbb{R}^3 \to \mathbb{R}^3$ connected with the identity sending K to K'. A knot parametrisation is a C^1 embedding $\gamma_K : S^1 \to K \subset R^3$.

6.1.1 Framing independence relation in chord diagrams and the algebra $\mathcal{A}'$

Consider the equivalence relation on the space of chord diagrams shown on figure 18. It is called framing independence relation.

A chord c like in figure 18, such that there are no extreme points of any other chord between its initial and end points, is called a isolated chord. Obviously, the sub vector space $1T$ of $\mathcal{A}_{\text{fin}}$ generated by the chord diagrams with isolated chords is an ideal of $\mathcal{A}_{\text{fin}}$, in fact it is the ideal $<\ominus>$ generated by the chord diagram with only one chord. Therefore $\mathcal{A}'_{\text{fin}} = \mathcal{A}_{\text{fin}}/1T$ is an abelian algebra. It has an obvious grading similar to the grading of $\mathcal{A}$, where the grading coefficient of a chord diagram is given its number of chords. Therefore $\mathcal{A}'$, by definition the graded completion of $\mathcal{A}'_{\text{fin}}$, is

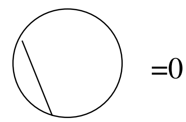

Figure 18: Framing Independence Relation

also an abelian algebra. This algebra is the target space for both the Kontsevich Integral and Unframed Kontsevich Universal Knot Invariant.

6.1.2 Unframed Kontsevich Universal Knot Invariant

As described for example in [BN] there exists a map $\mathcal{K} \xrightarrow{\mathcal{Z}_u} \mathcal{A}'$, which is called the Unframed Kontsevich Universal Knot Invariant. Let us describe how it is constructed out of the Kontsevich Integral. Another good reference for the description of the Kontsevich Integral is [CD], which we follow closely. To begin with we recall the definition of Morse functions. These are functions $f : S^1 \to \mathbb{R}$, such that:

1. f is C^1.

2. Each critical point of f is either a local maximum or a local minimum.

3. f attains different values at each extreme point.

4. The set of critical points of f is finite.

Consider a knot K and a parametrisation $\gamma : S^1 \to \mathbb{R}^3 \cong \mathbb{R} \times \mathbb{C}$ of it. Let $t(s)$ be the projection of $\gamma(s)$ in the first variable. Apart from possibly applying an isotopy of $\mathbb{R}^3$, we can suppose $t(s)$ is a Morse function. Let $\{t_0, ..., t_n\}$ be the set of critical points of t. Suppose $t_0 < t_i < ... < t_n$, and define $I_k = (t_{k-1}, t_k), k = 1, ..., n$. Given a positive integer m, define:

$$I_{k,m} = \{(x_1, ..., x_m) \in \mathbb{R}^m : t_{k-1} < x_1 < ... < x_m < t_k\}, k = 1, ..., n.$$

We consider also $I_{k,0} = \{0\}, k = 1, ..., n$. Therefore the volume of $I_{k,m}$ is $\frac{1}{m!}(t_k - t_{k-1})^m$. Suppose $m > 1$ is a integer. For any finite sequence $\underline{m} = (m_1, ..., m_n) \in \mathbb{N}_0^n$, define

$$I_{\underline{m}} = I_{1,m_1} \times ... \times I_{n,m_n}.$$

Let $k \in \{1, ..., n\}$. The intersection $K_t = (\{t\} \times \mathbb{C}) \cap K$ has a constant number of points in I_k. See figure 19. Obviously all the sets $K_t, t \in I_k$ can be identified with a finite set $S(k)$ in such a way the obvious map $K \cap (I_k \times \mathbb{C}) \to S(k)$ is continuous. Let $p \in S(k)$, we can define uniquely a map $z_p : I_k \to \mathbb{C}$ such that $(t, z_p(t)) = p, \forall t \in I_k$, by the implicit function theorem z_p is C^1. It can be extended obviously to a continuous function $z_p : [t_{k-1}, t_k] \to \mathbb{C}$.

Denote by $S^{(2)}(k)$ the set of subsets of $S(k)$ with 2 elements. We call them chords. There exists a subset $D_+(k) \subset S^{(2)}(k)$, the set of elements $\{p, q\} \in S^{(2)}(k)$ for which $(t_k, z_p(t_k)), (t_k, z_q(t_k))$ are both local maxima. The set $D_-(k)$ is defined analogously as the set $\{p, q\} \in S^{(2)}(k)$ for each $(t_{k-1}, z_p(t_{k-1})), (t_k, z_q(t_k))$ are both local minima. Notice that each set $D_+(k), D_-(k)$ has cardinality at most one by the condition 4 of the definition of Morse functions.

Let us be given an integer $m > 0$ and a $k \in \{1, ..., n\}$. We associate to them a finite set $\mathcal{P}_m^k$. An element of $\mathcal{P}_m^k$ is a map $P^k : \{1, ..., m\} \to S^{(2)}(k)$ of the form $i \mapsto \{p_k(i), p'_k(i)\}$, equivalently a map $i \to \{{z^k}_i(t), {z'^k}_i(t)\}, t \in I_k$. The map P^k is called a pairing and each $\{z^k, z'^k\}$ a chord of it. We put the condition that $\{p_k(m), p'_k(m)\} \notin D_+(k)$ and $\{p_k(1), p'_k(1)\} \notin D_-(k)$.

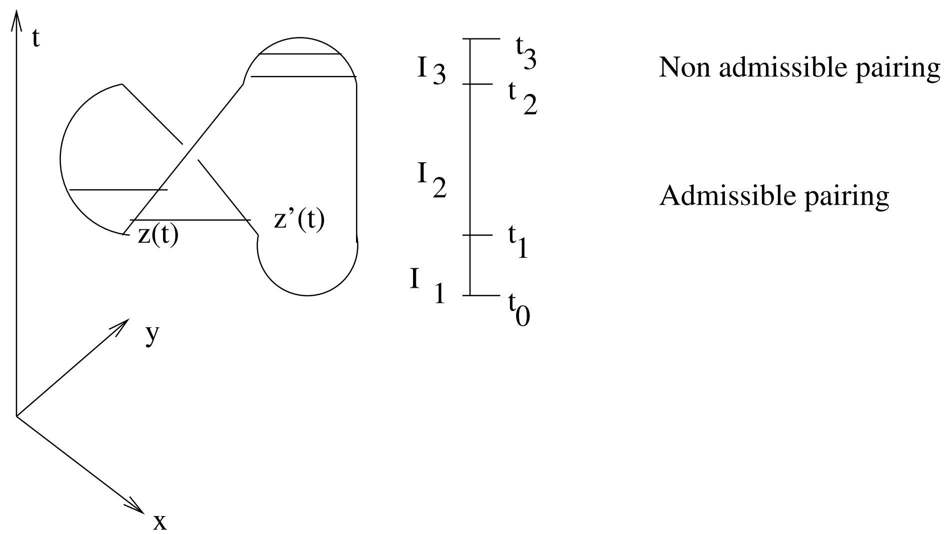

Figure 19: Admissible and Non Admissible pairings

Definition 58 *An element of $P^k \in \mathcal{P}^k_m$ is called a k-admissible pairing. See figure 19*

Proposition 59 *Given an admissible pairing $P^k = \{i \mapsto \{z^k{}_i(t), z'^k{}_i(t)\}\} \in \mathcal{P}^k_m$, the following integral is absolutely convergent:*

$$Z(k, m, P^k) = \int_{I_{k,m}} \bigwedge_{i=1}^{m} \frac{dz^k_i(x_i) - dz'^k_i(x_i)}{z^k_i(x_i) - z'^k_i(x_i)}. \tag{90}$$

This will the basis for the definition of the Kontsevich integral.

Definition 60 *Suppose we are given a sequence $\underline{m} = (m_1, ..., m_n) \in \mathbb{N}_0{}^n$, an $\underline{m}$-pairing P (admissible) is a map*

$$P \in \{1, ..., n\} \to P^k \in \bigsqcup_{k=1}^{n} \mathcal{P}^k_{m_k},$$

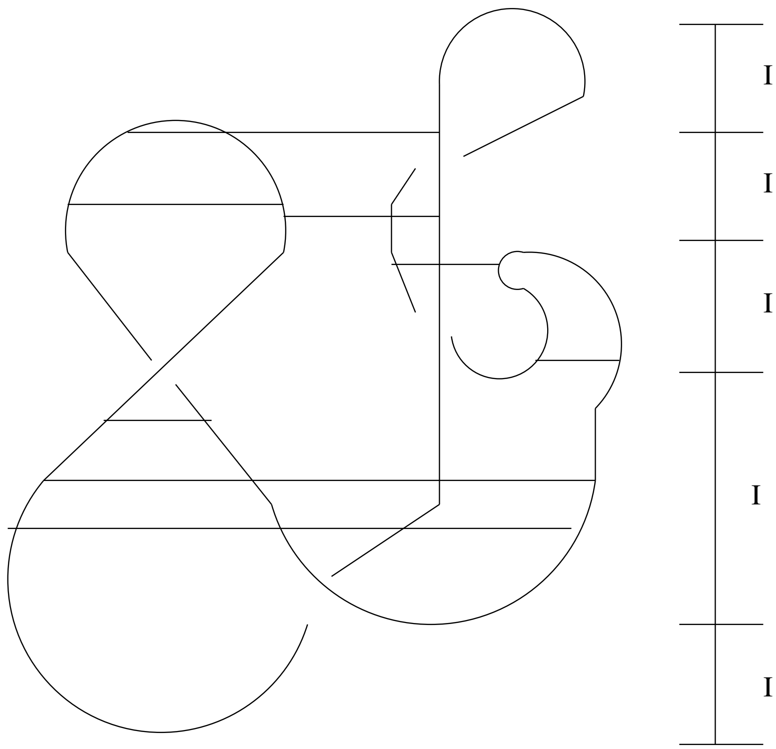

Figure 20: A $(0,3,2,3,0)$-pairing

with $P^k \in \mathcal{P}^k_{m_k}, k = 1, ...n$ *a* k *admissible pairing. Let* $\mathcal{P}_{\underline{m}}$ *designate the set of all* $\underline{m}$*-pairings. Any pairing* P *defines a chord diagram* ω_P *in the obvious way. Let for any* $P \in \mathcal{P}_{\underline{m}}$*:*

$$Z(\underline{m}, P) = \prod_{k=1}^{n} Z(k, m_k, P(k)).$$

The reader is advised to look at figure 20. Notice there are no pairings with m_1 or m_5 not equal to zero, for then the pairings P^0 and P^5 would not be admissible.

One more simple ingredient is necessary to define the Kontsevich universal knot invariant. Let us be given a pairing P, then $\#P$ is defined as the number of z_i^k or

${z'}_i^k$ on which $t(s)$ is decreasing.

Let $\underline{m} = (m_0, ..., m_n) \in \mathbb{N}_0^n$, we put $|\underline{m}| = m_0 + ... + m_m$.

Definition 61 *Consider a C^∞ parametrisation $\gamma : S^1 \to \mathbb{R} \times \mathbb{C}$ such that the projection $t(s)$ in the first variable is a Morse function with $n+1$ critical points. Define for every integer $m > 1$:*

$$\mathbf{Z}(\gamma, m) = \sum_{\substack{\{\underline{m} \in \mathbb{N}_0{}^n : |m| = m\} \\ P \in \mathcal{P}_{\underline{m}}}} (-1)^{\#P} Z(\underline{m}, P) w_P.$$

Define also $\mathbf{Z}(\gamma, 0) = 1, \mathbf{Z}(\gamma, 1) = 0$. The Kontsevich integral is by definition:

$$\mathbf{Z}(\gamma) = \sum_{m \in \mathbb{N}_0} \frac{1}{(2\pi i)^m} \mathbf{Z}(\gamma, m) \in \mathcal{A}'.$$

Consider a parametrisation γ_∞ of the unknot such that $t(s)$ is a Morse function having 4 critical points. The next two results are proved for example in [BN]

Theorem 62 *The series $\mathbf{Z}(\gamma)$ does not depend on the chosen parametrisation γ_∞ yielding the unknot as long as $t(s)$ is a Morse function which have exactly 4 critical points. Call the series obtained $\mathbf{Z}(\infty)$.*

Notice that $\mathbf{Z}(\infty)$ is invertible in $\mathcal{A}'$ for its first term is 1.

Theorem 63 *Let us be given a knot K, with a parametrisation γ_K of it such that $t(s)$ is a Morse function with $n+1$ critical points. Notice $n+1$ is a positive even number. Define*

$$\mathcal{Z}_u(K) = \frac{\mathbf{Z}(\gamma_K)}{\mathbf{Z}(\infty)^{\frac{n+1}{2}}}$$

Then $\mathcal{Z}_u(K)$ does not depend on the isotopy class of of K, as long as $t(s)$ is a Morse function, which we can suppose apart from isotopy. Therefore $\mathcal{Z}_u(K)$ is a knot invariant. Call it the unframed Kontsevich Universal Knot invariant.

6.1.3 Framed Kontsevich Universal Knot Invariant

We now extend the Kontsevich Universal Knot Invariant for framed knots. This extension is referred to usually as the framed Kontsevich Universal Knot Invariant.

Recall that a framed knot K is an embedding of the stripe $S^1 \times [-1, 1]$ into R^3 with the obvious notion of isotopy class. Any framed knot defines an isotopy class of knots in the obvious way. The framing coefficient $F(K)$ of a framed knot K is the linking number of $K \times \{-1\}$ and $K \times \{1\}$.

Recall that the algebra $\mathcal{A}'$ is defined as the graded completion of $\mathcal{A}'_{\text{fin}} = \mathcal{A}_{\text{fin}}/ < \ominus >$, where $< \ominus >$ is the ideal of $\mathcal{A}_{\text{fin}}$ generated by the chord diagram with only one chord. Both $\mathcal{A}_{\text{fin}}$ and $\mathcal{A}'_{\text{fin}}$ have gradings for which the grading coefficient $\deg(\omega)$ of a chord diagram is given by its number of chords. Consider the map $\phi : \mathcal{A}_{\text{fin}} \otimes \mathcal{A}_{\text{fin}} \to \mathcal{A}_{\text{fin}}$ such that if ω_1 and w_2 are chord diagrams we have $\phi(\omega_1 \otimes \omega_2) = (-\ominus)^{\deg(\omega_1)} w_2$. Define $\psi : \mathcal{A}_{\text{fin}} \to \mathcal{A}_{\text{fin}}$ as $\psi = \phi \circ \Delta$. Then actually $\psi^2 = \psi$ and ψ is a Hopf algebra morphism (this is a non trivial fact). Its kernel is contained in $< \ominus >$, thus ψ descends to a map $\psi' : \mathcal{A}'_{\text{fin}} \to \mathcal{A}_{\text{fin}}$, which can be extended to all $\mathcal{A}'$ since ψ' preserves gradings. See [Wi2] for more details. Let $p : \mathcal{A}' \to \mathcal{A}$ be the projection. Given that $\psi(\ominus) = 0$ it follows

$$\psi' \circ p = \psi. \tag{91}$$

Definition 64 *The (framed) Kontsevich Universal Knot Invariant is:*

$$\mathcal{Z}_f(K) = \mathcal{Z}(K) = \psi'(\mathcal{Z}_u(K)) \exp(F(K)\ominus),$$

where the convergence of the series defining $e^{(F(K)\ominus)}$ *is understood in the graded algebra sense. Here* $\mathcal{Z}_u(K)$ *denotes the unframed Kontsevich Universal Knot Invariant. Recall* $\mathcal{F}(K)$ *is the framing coefficient of the knot* K*.*

Recall the Kontsevich Universal Knot Invariant satisfies all the properties stated in theorem 1.

This is a slightly *ad-hoc* definition of the framed Kontsevich Universal knot invariant. However it is equivalent to the standard one. This was pointed out in [Wi1]. The proof of this fact appears in [LM2](theorem 5.13). Let us give an idea of how this result can be proved. I only know one place where this result is proved, namely [LM2].

Proof. **(sketch)** We can either define the Framed Kontsevich Universal Knot Invariant $\mathcal{Z}'$ out of the Drinfeld Associator as in [LM2]; or by regularising the integral (90) for non admissibe pairings through considering

$$\lim_{\varepsilon \to 0, \varepsilon > 0} \mathcal{Z}(K_e) \exp\left(\frac{-\ln(\varepsilon)}{2\pi i}\ominus\right),$$

at each minimal point, say, where K_ε is the tangle obtained from the knot K by removing the points in the two strands making the minimal point which are at smaller distance than ε. We do analogously for maximal points considering

$$\lim_{\varepsilon \to 0, \varepsilon > 0} \mathcal{Z}(K_e) \exp\left(\frac{\ln(\varepsilon)}{2\pi i}\ominus\right).$$

At the end we obviosly consider the limit for all maximal and minimal points simultaneously. See [LM2] for more details. Notice that both definitions of the Framed

Kontsevich Universal Knot invariant yield isotopy invariants of blackboard framed knots. This is because we are implicitly choosing determination of the logarithm at each extreme point. Call these two definitions $\mathcal{Z}_D$ and $\mathcal{Z}_R$. It is well known that they are equivalent, that is $\mathcal{Z}' = \mathcal{Z}_R = \mathcal{Z}_D$, see [LM2] or [AF]. Let us prove both are equal to $\mathcal{Z}$, as in Definition 64.

Consider the algebra projection $p : \mathcal{A} \to \mathcal{A}'$. From the second definition of $\mathcal{Z}$, the one involving the regularisation of the Kontsevich Integral, it is immediate that $p \circ \mathcal{Z}' = \mathcal{Z}_u$. Therefore $\psi' \circ \mathcal{Z}_u = \psi' \circ p \circ \mathcal{Z}' = \psi(\mathcal{Z})$, by (91). Therefore, we only need to find the relation between $\psi \circ \mathcal{Z}'$ and $\mathcal{Z}'$. It is now better to consider $\mathcal{Z}' = \mathcal{Z}_D$, the definition of the Kontsevich Universal Knot Invariant involving the Drinfeld associator as in [LM1]. Let $\phi(A, B)$ be the Drinfeld associator, constructed from the monodromy of the Kniznick-Zamolodchikov equations. See [D1], [D2], [K] or [LM3]. It is an element of the algebra $\mathbb{C}[[A, B]]$ of power series in two non commuting variables A and B. We now follow [LM3] and [K]. Consider the algebra $\mathbb{C}[[A, B]]][\alpha, \beta]$ of polynomials in α, β with coefficients in $\mathbb{C}[[A, B]]$ (we suppose α and β commute with all other variables). A beautiful identity satisfied by $\phi(A, B)$ is

$$\phi(A - \alpha, B - \beta) = \phi(A, B), \text{ in } \mathbb{C}[[A, B]][\alpha, \beta]$$

Therefore if c commutes with all variables we have

$$\phi(A - c, B - c) = \phi(A, B), \text{ in } \mathbb{C}[[A, B]][c]] \tag{92}$$

see [LM1] page 63 or [K] page 469. A proof of this fact is not that difficult, but involves going through all the definition of $\phi(A, B)$. Equation 92 is more or less obvious from the fact $\phi(A, B)$ is an exponential of a series of commutators in A and

B.

Let K be a framed knot. It is better now to see $\psi : \mathcal{A} \to \mathcal{A}$ as a map such that if ω is a chord then $\psi(\omega) = w - b_1/2 - b_2/2$, where b_1 (resp b_2) is a chord connecting two points in a neighbourhood of one (resp other) extreme point of w; and analogously for more complicated chord diagrams. The advantage is that it makes also sense to consider $\psi(\omega)$ where ω is a chord diagram supported on a tangle. The evaluation of $\mathcal{Z}_D(K)$ is made out of terms like $\phi(H_{i,j}, H_{i,j})$, where $H_{i,j}$ means connecting the i^{th} and j^{th} strands of K by a chord perpendicular to the t axis. We also have terms like $\exp(H_{i,j})$ at each crossing point. Consider applying ψ to $\mathcal{Z}_D(K)$. The terms involving $\phi(H_{i,j}, H_{j,k})$ are easy to deal with: by 92 they remain unaltered since $\ominus$ commutes with any chord diagram. Since ψ is an algebra morphism, the evaluation of $\exp(H_{i,j})$ will change as $\exp(H_{i,j} - b_1/2 - b_2/2)$, which, since $\ominus$ commute with all variables, is $\exp(H_{i,j})\exp(-\ominus)$. This is the reason why the factor $\exp(F(K))$ in Definition 64 appears. We thus need to have $\mathcal{Z} = \mathcal{Z}'$. See [LM2] for details. ∎

6.1.4 Some bounds for the coefficients of Kontsevich Universal Knot Invariant

We now *estimate* the modulus of the coefficients of the Kontsevich integral for knots. This will be used in the proof of theorem 34. We freely use the notation of 6.1.2. This will be a rather long section, not very difficult, but with a large amount of notation being introduced.

Let K be a knot and let $\gamma : S^1 \to \mathbb{R} \times \mathbb{C}$ be a Morse parametrisation of it. To calculate $\mathbf{Z}(\gamma)$, we need to know the critical points $t_0 < t_1 < ... < t_n$ of $t(s)$ of γ_K. If $I_k = [t_{k+1}, t_k]$ then $I_1 \cup ... \cup I_n$ is a splitting $[t_0, t_n] = \gamma_K(S^1)$ such that no critical point of γ_K is in the interior of any interval I_k. However to calculate $\mathbf{Z}(\gamma)$, we can use any splitting $\tau_0 = c_0 < c_1 < .... < c_N = t_n$ of the interval $[t_0, t_n]$ as long as the set of critical points of $t(s)$ is contained in $\{c_0, ..., c_N\}$. See [BN], 4.3.2. This is a consequence of the fact that away from extreme points the Kontsevich Integral is expressed as the holonomy of the formal Kniznick-Zamolodchikov connection. See [BN]

Let K be an oriented knot, it can be seen as the closure of some braid $b \in B(n)$ for some $n \in \mathbb{N}$. Here $B(n)$ is the braid group with n-strands. A parametrisation γ_K of K will have therefore $2n$ critical points. Fix once for all a braid $b \in B(n)$, for some $n \in \mathbb{N}$ such that its closure is K. We want to calculate $\mathbf{Z}(K)$ out of the closure of b, thus for definiteness we explicitely construct such a closure. The braid b consists of n oriented downwards strands connecting the points $(1,0), (1,1), ..., (1, n-1)$ to $(-1,0), (-1,1)..., (-1,n)$. Strands which can be supposed contained in $[-1,1] \times \{z \in \mathbb{C} : |z| < n\}$. Consider an explicit construction of a closure of b as in figure 21. That is:

Consider n straight lines oriented upwards connecting

- $(n+1, 1)$ to $(n+1, -1)$
- $(n+2, 1+1)$ to $(n+2, -(1+1))$
- $(n+3, 1+1+2)$ to $(n+3, -(1+1+2))$
- $(n+4, 1+1+2+3)$ to $(n+4, -(1+1+2+3))$
- and so on until
- $(n+n, 1+1+2+3+...+n-1)$ to $(n+n, -(1+1+2+3+...+n-1))$

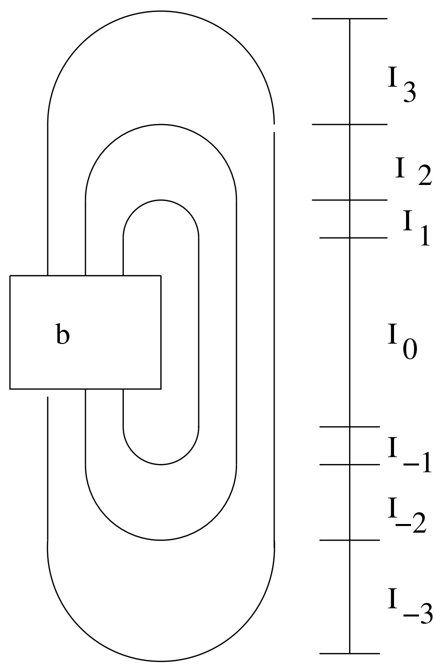

Figure 21: Closure of a braid

Also, connect the points

- $(n-1,1)$ to $(n-1,1)$
- $(n-2,1)$ to $(n-2,1+1)$
- $(n-3,1)$ to $(n-3,1+1+2)$
- $(n-4,1)$ to $(n-4,1+1+2+3)$
- and so on until
- $(n-n,1)$ to $(n-n,1+1+2+3+...+n-1)$

By oriented downwards straight lines.

Similarly, use oriented downwards straight lines, and connect the points

- $(n-1,-1)$ to $(n-1,-1)$
- $(n-2,-1)$ to $(n-2,-(1+1))$
- $(n-3,-1)$ to $(n-3,-(1+1+2))$
- $(n-4,-1)$ to $(n-4,-(1+1+2+3))$
- and so on until
- $(n-n,-1)$ to $(n-n,-(1+1+2+3+...+n-1))$

Now, use semicircles of radius k oriented counterclockwise, chosen so that the closure of the braid has a C^1 parametrisation, to connect the points

- $(n+1, 1)$ to $(n-1, 1)$
- $(n+2, 1+1)$ to $(n-2, 1+1)$
- $(n+3, 1+1+2)$ to $(n-3, 1+1+2)$
- $(n+4, 1+1+2+3)$ to $(n-4, 1+1+2+3)$
- and so on until
- $(n+n, 1+1+2+3+...+n-1)$ to $(n-n, 1+1+2+3+...+n-1)$

And, similarly

- $(n+1, -1)$ to $(n-1, -1)$
- $(n+2, -(1+1))$ to $(n-2, -(1+1))$
- $(n+3, -(1+1+2))$ to $(n-3, (1+1+2))$
- $(n+4, -(1+1+2+3))$ to $(n-4, -(1+1+2+3))$
- and so on until
- $(n+n, -(1+1+2+3+...+n-1))$ to $(n-n, -(1+1+2+3+...+n-1))$

The reader is strongly advised to look at figure 21. Let $\gamma : S^1 \to \mathbb{R} \times \mathbb{C}$ be the parametrisation of K just constructed. Define $r_0 = 1, r_1 = 1+1, r_3 = 1+1+2, ..., r_n = 1+1+2+3+...+n$. Split $\gamma(S^1) = [-r_n, r_n]$ as $[-r_n, -r_{n-1}] \cup ... \cup [-r_2, -r_1] \cup [-r_1, r_1] \cup [r_1, r_2] \cup ... \cup [r_{n-1}, r_n] = I_{-n} \cup I_{-n-1} \cup ... \cup I_{-1} \cup I_0 \cup I_1 \cup ... \cup I_n$

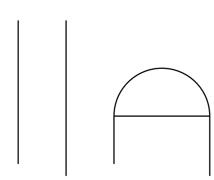

Figure 22: A chord of type A

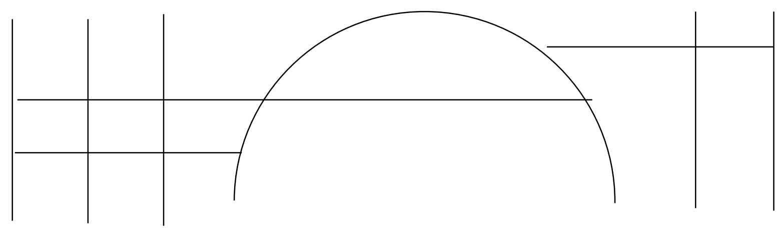

Figure 23: Chords of type B_1

Definition 65 (Chord Type) *Let $P^k \in \mathcal{P}^k_m$ with $k \in \{-n, ..., -1\} \cup \{1, ..., n\}$, be a k-admissible pairing with m chords, in other words a map $i \mapsto p_i = \{z_i, z_i'\}$, where $i \in \{1, ..., m\}$ and $z, z' : I_k \to \mathbb{C}$. We divide the chords $p_i = \{z_i, z_i'\}$ into two different types: A chord is said to be of type A, say, if it looks like the chord in figure 22. We say it is of type B otherwise. Chords of type B can appear on types B_1 and B_2 shown in figures 23 and 24. If $k = 0$ then by definition we say that all the chords are of type B_2. If P^k is a k-admissible pairing then $A(P^k)$ (resp $B(P^k)$) is the number of chords of P of type A (resp B), thus $A(P^k) + B(P^k) = m$, which is the number of chords of P^k. For a chord p, let $T(p)$ denote its type.*

Definition 66 (Type Sequences) *Let b be a braid with n strands whose closure is a knot and $k \leq 0$. Let also $P^k \in \mathcal{P}^k_m$ be a k-admissible pairing with m-chords.*

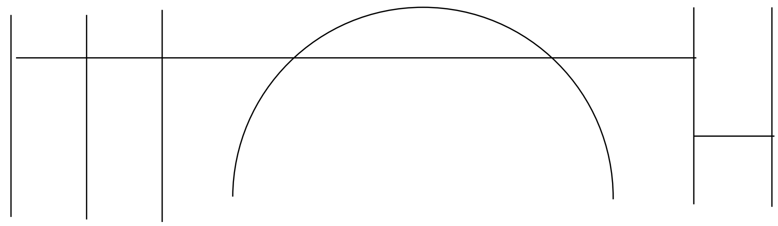

Figure 24: chords of type B_2

The type sequence of P^k *is by definition*

$$\mathcal{T}(P^k) = (T(p_1), T(p_2), ..., T(P_m)).$$

If $k > 0$ *then by definition*

$$\mathcal{T}(P^k) = (T(p_m), ..., T(p_2), T(p_1)).$$

Notice that since the pairings are admissible, the first letter of the type sequences is always B*. Let* $\underline{m} = (m_{-n}, ..., m_0, ..., m_m)$ *be a sequence in* $\mathbb{N}_0^{2n+1}$*. Let also* $P \in \mathcal{P}_{\underline{m}}$ *be a* $\underline{m}$ *pairing, thus for each* $k \in \{-n..., n\}$*, we have type sequences* $\mathcal{T}(P^k)$ *characterising the pairing* $P^k \in \mathcal{P}^k_{m_k}$*. Let* $B(P)$ *and* $A(P)$ *be the number of chords of* P *of type* B *and* A*, respectively, thus* $A(P) = A(P^{-n}) + ... + A(P^n)$*, and similarly for* $B(P)$

The reader is advised to look at figures 25 and 26.

Consider a pairing in $\mathcal{P}^k_m$ with $k \neq n$. Due to the way we defined the closure of the braid b, it is very easy to determine the functions $z_p : I_n \to \mathbb{C}$. Only two of them will be non constant. If k is positive, then they are the obvious shifting of $\pm\sqrt{k^2 - t^2}$, and analogously if k is negative.

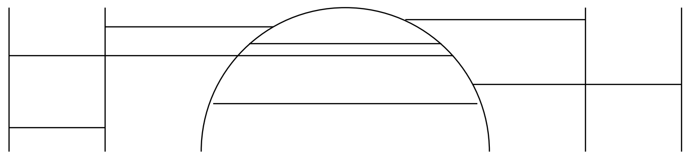

Figure 25: An example with $\mathcal{T}(P) = (B, B, A, B, B, B, B, A, B)$

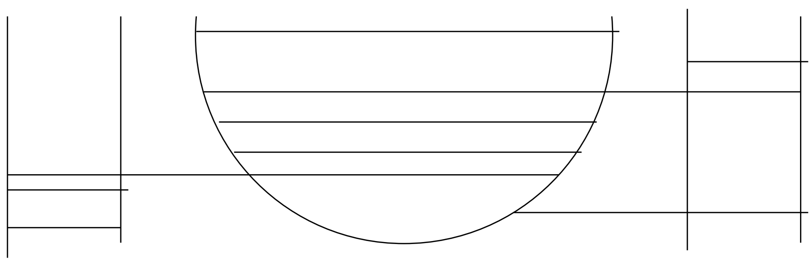

Figure 26: An example with $\mathcal{T}(P) = (B, B, B, B, A, A, B, B, A)$

Lemma 67 *Let us be given a* $k \in \{-n, ..., -1\} \cup \{1, ..., n\}$*,* $m \in \mathbb{N}$ *and a* k*-admissible pairing* $P^k = \{i \mapsto \{z_i^k(t), z_i'^k(t)\}\} \in \mathcal{P}_m^k$*. We have the estimate*

$$|Z(k, m, P^k)| = \left| \int_{I_{k,m}} \bigwedge_{i=1}^{m} \frac{dz_i^k(x_i) - dz'^k_i(x_i)}{z_i^k(x_i) - z'^k_i(x_i)} \right|$$
$$\leq \frac{k^{-B(P^k)}}{B(P^k)!} \prod_{\substack{i \in \{1,...,m\} \\ \mathcal{T}(P^k)(i) = A}} \frac{1}{\#\{j \in \{1, ..., i\} : \mathcal{T}(P^k)(j) = B\}},$$

where $\mathcal{T}(P^k) = (\mathcal{T}(P^k)(1), ..., \mathcal{T}(P^k)(m))$ *is the type sequence of* P^k*. Here* $\#$ *denotes the cardinal of a set.*

Proof. We make the proof for $k \in \{1, ..., n\}$, the other proof will be analogous. Obviously:

$$|Z(k, m, P^k)| = \int_{0 < x_1 < x_2 < ... < x_m < k} f_1(x_1)...f_m(x_m) dx_1...dx_m.$$

Here $f_i(t)$ can be a function of the kinds: $B^{(l)}, l > k$, where

$$B^{(l)}(t) = \left| \frac{\frac{d}{dt}\left(l \pm \sqrt{k^2 - t^2}\right)}{l \pm \sqrt{k^2 - t^2}} \right| = \left| \frac{1}{l \pm \sqrt{k^2 - t^2}} \frac{t}{\sqrt{k^2 - t^2}} \right| < \frac{t}{\sqrt{k^2 - t^2}}, l > k,$$

if the associated chord $P^k(i)$ is of type B_1; $A(t)$, where

$$A(t) = \frac{t}{k^2 - t^2} = \left| \frac{\frac{d}{dt}\sqrt{k^2 - t^2}}{\sqrt{k^2 - t^2}} \right|,$$

if the associated chord $P^k(i)$ is of type A; or the null function if the associated chord $P^k(i)$ is of type B_2. Since we are considering admissible pairings, we also have $f_m \neq A$. This is a crucial condition for the integral to be convergent in the first place. Let $a_i = 1/2$ if the i^{th} chord is of type B and 1 if the i^{th} chord is of type A, thus $a_m = 1/2$. We have:

$$|Z(k, m, P^k)| \leq \int_0^k dx_1 \frac{x_1}{(k^2 - x_1^2)^{a_1}} \int_{x_1}^k dx_2 \frac{x_2}{(k^2 - x_2^2)^{a_2}} \int_{x_2}^k dx_3 \frac{x_3}{(k^2 - x_3^2)^{a_3}} \int_{x_3}^k dx_4 ... \int_{x_{m-1}}^k dx_m \frac{x_m}{\sqrt{k^2 - x_m^2}} = \mathcal{I}.$$

Consider the coordinate change $x_i = kx'_i$, we obtain:

$$\mathcal{I} = k^{-B(P^k)} \int_0^1 dx'_1 \frac{x_1}{(1 - {x'_1}^2)^{a_1}} \int_{x'_1}^1 dx'_2 \frac{x'_2}{(1 - x'^2_2)^{a_2}} \int_{x'_2}^1 dx'_3 \frac{x'_3}{(1 - x'^2_3)^{a_3}} \int_{x'_3}^1 dx_4 ... \int_{x'_{m-1}}^1 dx'_m \frac{x'_m}{\sqrt{1 - x'^2_m}}.$$

Thus if we put $x'_i = \sqrt{1 - y_i}, i = 1, ..., m$ yields:

$$\begin{aligned}
\mathcal{I} &= \frac{k^{-B(P^k)}}{2^m} \int_0^1 dy_1 \frac{1}{y_1^{a_1}} \int_0^{y_1} dy_2 \frac{1}{y_2^{a_2}} \int_0^{y_2} dy_3 \frac{1}{y_3^{a_3}} \int_0^{y_3} dy_4 ... \int_0^{y_{m-1}} dy_m \frac{1}{\sqrt{y_m}} \\
&= \frac{k^{-B(P^k)}}{2^m} \int_0^1 dy_m \frac{1}{\sqrt{y_m}} \int_{y_m}^1 dy_{m-1} \frac{1}{y_{m-1}^{a_{m-1}}} \int_{y_{m-1}}^1 dy_{m-2} \frac{1}{y_{m-2}^{a_{m-2}}} \int_{y_{m-2}}^1 dy_{m-3} ... \int_{y_2}^1 dy_1 \frac{1}{{y_1}^{a_1}}
\end{aligned}$$

Therefore we can use Corollary 75. The case $k < 0$ is analogous but missing the last step. ■

The behaviour of $Z(k, m, P^k)$ for $k = 0$ is easy to determine given that we have no infinities in the domains of integration. Obviously, there exists a constant $C < \infty$ such that:

$$\left| \frac{\frac{d}{dt}\left(z(t) - z'(t)\right)}{z(t) - z'(t)} \right| \leq C,$$

for any chord $z(t) - z'(t)$. In particular, given any 0-admissible pairing $P^0 \in \mathcal{P}^0_m$ we have

$$|Z(0, m, P)| < \frac{(2C)^m}{m!}.$$

Notice in this case $\mathcal{T}(P^0) = (B, B, .., B)$, by definition. Therefore:

Lemma 68 *Let b be a braid with n strands whose closure is a knot. There exists a constant $D < \infty$ such that for any $k \in \{-n, ..., , n\}$, any $m \in \mathbb{N}$, and any k-admissible pairing $P^k = i \mapsto \{z^k_i(t), z'^k_i(t)\} \in \mathcal{P}^k_m$ we have the estimate*

$$|Z(k, m, P^k)| \leq \frac{C^m}{B(P^k)!} \prod_{\substack{i \in \{1, ..., m\} \\ \mathcal{T}(P^k)(i) = A}} \frac{1}{\#\{j \in \{1, ..., i\} : \mathcal{T}(P^k)(j) = B\}}.$$

This will be the main theorem of this paragraph:

Theorem 69 *Let b be a braid whose closure is a knot. There exists a constant $D < +\infty$ such that for any sequence $\underline{m} \in \mathbb{N}_0^{2n+1}$ and pairing $P \in \mathcal{P}_{\underline{m}}$ we have:*

$$|Z(\underline{m}, P)| \leq \frac{D^{|\underline{m}|}}{B(P)!} \prod_{k=-n}^{n} \prod_{\substack{i \in \{1, ..., m_k\} \\ T(P^k)(i) = A}} \frac{1}{\#\{j \in \{1, ..., i\} : \mathcal{T}(P^k)(j) = B\}}$$

where $\underline{m} = (m_{-n}, ..., m_n)$ and $|\underline{m}| = m_{-n} + ... + m_n$.

Proof. This follows from Proposition 68 and Lemma 70. ■

6.1.5 Some calculations

We prove the lemmas that we have just used used, as well as some results that will be useful later. First of all recall Stirling's inequality for $n!$:

$$\sqrt{2\pi m}\frac{m^m}{e^m}e^{\frac{1}{12m+1}} < n! < \sqrt{2\pi m}\frac{m^m}{e^m}e^{\frac{1}{12m}}, m = 1, 2, ... \tag{93}$$

It will be used quite frequently later. Immediately to prove:

Lemma 70 *Let n be an integer. There exists a $C > 0$ such that for all sequences $\underline{m} = (m_1, ..., m_n) \in \mathbb{N}_0^n$ we have*

$$\frac{|\underline{m}|!}{(m_1!)(m_2!)...(m_n!)} \leq C^{|\underline{m}|},$$

where as usual $|\underline{m}| = m_1 + m_2 + ... + m_n$.

Proof. Let $m = |\underline{m}|$. Suppose $m_i \neq 0, \forall i$. The Stirling inequalities prove:

$$\frac{m!}{(m_1!)(m_2!)...(m_n!)} < \frac{m^m}{(m_1^{m_1})(m_2^{m_2})...(m_n^{m_n})}\frac{\sqrt{2\pi}\sqrt{m}}{\sqrt{2\pi}^n\sqrt{m_1...m_n}}$$
$$\frac{e^{\frac{1}{12m_1+1}}e^{\frac{1}{12m_2+1}}...e^{\frac{1}{12m_n+1}}}{e^{\frac{1}{12m}}}.$$

Therefore for any $\underline{m} \in \mathbb{N}^n$:

$$\frac{m!}{(m_1!)(m_2!)...(m_n!)} < \frac{m^m}{(m_1^{m_1})(m_2^{m_2})...(m_n^{m_n})}\frac{\sqrt{2\pi}\sqrt{m}}{\sqrt{2\pi}^n}e^n.$$

We can thus choose a $D > 0$ such that:

$$\frac{m!}{(m_1!)(m_2!)...(m_n!)} < \frac{m^m}{(m_1^{m_1})(m_2^{m_2})...(m_n^{m_n})}D^m,$$

for all $\underline{m} \in \mathbb{N}^n$.

The minimum of $(m_1^{m_1})(m_2^{m_2})...(m_n^{m_n})$ with the restriction $m_1+m_2+...+m_n = m$ is attained when $m_i = m/n, i = 1, 2, ..., n$. A simple Langrange multiplier calculation shows this. In particular

$$\frac{m!}{(m_1!)(m_2!)...(m_n!)} < \frac{m^m}{\left(\frac{m}{n}\right)^m} D^m \leq C^m,$$

for any $m_1, ..., m_n \in \{1, 2, ...\}$, for some fixed constant C. The case in which we admit some m_i's to be zero obviously reduces to this one for a smaller n. ■

The following two simpler lemmas are also useful:

Lemma 71 *Let n be an integer. There exists a $C > 0$ such that*

$$(n+m)! \leq m!C^m, \forall m \in \mathbb{N}_0.$$

Proof. Obviously $\frac{(n+m)!}{m!} < (n+m)^n$ (we are not using Stirling Inequalities). This term has polynomial growth with respect to m, thus $\frac{(n+m)!}{m!} < C^m$ for some fixed C not depending on m.

■

Lemma 72 *There exist constants $B, C < \infty$ such that*

$$B^n n^n < n! < C^n n^n, n = 1, 2, ...$$

Proof. This is essentially obvious from Stirling Inequalities ■

Now, a very important set of calculations:

Lemma 73 *Consider an integral of the form:*

$$\mathcal{I}(a_1, a_2, ..., a_n) = \int_{-\infty < x_1 < x_2 < ... < x_n < 0} e^{a_1 x_1 + a_2 x_2 + ... + a_n x_n} dx_1 dx_2 ... dx_n,$$

where $a_1 > 0, a_1 + a_2 > 0, ..., a_1 + a_2 + ... + a_n > 0$ *(the case in which some* a_i'*s may be equal to* 0 *is admitted) . We have:*

$$\mathcal{I}(a_1, a_2, ..., a_n) = \frac{1}{a_1(a_1 + a_2)...(a_1 + a_2 + ... + a_n)}.$$

Proof. We proceed by induction in n. This is trivial for $n = 1$ and $n = 2$. Suppose $a_n \neq 0$, a simple integration by parts in the variable x_n tells you:

$$\mathcal{I}(a_1, a_2, ..., a_n) = \frac{\mathcal{I}(a_1, a_2, ..., a_{n-1}) - \mathcal{I}(a_1, a_2, ..., a_{n-1} + a_n)}{a_n},$$

thus

$$\mathcal{I}(a_1, a_2, ..., a_n) = \frac{1}{a_1(a_1 + a_2)...(a_1 + a_2 + ... + a_n)}, \quad \text{if } a_n \neq 0.$$

The case $a_n = 0$ follows from continuity (we can do this because $a_1 > 0$). ■

We thus have:

Lemma 74 *Let* $T = (B, C_2, ..., C_m)$ *be a sequence of A's and B's starting in B. Let* $l_A = 0$, $l_B = 1$ *and* $B(T)$ *be the number of elements of* T *which are equal to* B*. Consider an integral of the form:*

$$\mathcal{I}(T) = \int_{-\infty < x_1 < x_2 < ... < x_n < 0} e^{l_{T_1} x_1 + l_{T_2} x_2 + ... + l_{T_n} x_n} dx_1 dx_2 ... dx_n.$$

Then

$$\mathcal{I}(T) = \frac{1}{B(T)!} \prod_{\substack{i \in \{1,...,m\} \\ T_i = A}} \frac{1}{\#\{j \in \{1, ..., i\} : T_j = B\}}.$$

Therefore:

Corollary 75 *Let $T = (B, ...)$ be a sequence of A's and B's of length m starting in the letter B. Let $B(T)$ be the number of B's in it. Let $c_i = 1/2$ if $T_i = B$ and $c_i = 1$ if $T_i = A$, for $i = 1, ..., m$. Then*

$$I(T) = \int_{0<x_1<...<x_n<1} \prod_{i=1}^{n} \frac{1}{x_i^{c_i}} dx_1...dx_m = 2^m \frac{1}{B(T)!} \prod_{\substack{i\in\{1,...,m\} \\ T_i = A}} \frac{1}{\#\{j \in \{1, ..., i\} : T_j = B\}}.$$

Proof. We just need to consider the change of variables $x_i = \exp(2t_i), i = 1, ..., m$ and apply the previous lemma. ■

6.2 Proof of Theorem 34

Consider the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ with its usual Cartan decomposition. The elements H, E, F defined in 1.4.6 are a basis of it. The infinitesimal R-matrix coming from minus the Cartan-Killing form in $\mathfrak{sl}(2, \mathbb{C})$ expresses as

$$t = -\frac{1}{4}\left(E \otimes F + F \otimes E + \frac{H \otimes H}{2}\right).$$

Let z be a complex number, the highest weight representation of maximal weight $2z$ is the representation $\overset{z}{\rho}$ of spin z defined in 1.4.6. For the case $z \notin \frac{1}{2}\mathbb{N}_0$, it is defined in the infinite dimensional space $\overset{z}{V}$ which has a basis of the form $\{v_{2z}, v_{2z-1}, v_{2z-2}, ...\}$. The action of H, E and F is:

$$Hv_k = (k - z)v_k; k = 2z, 2z - 1, ...,$$

$$Ev_k = (2z-k)v_{k+1}; k = 2z, 2z-1, ...,$$

$$Fv_k = kv_{k-1}; k = 2z, 2z-1, ...$$

If $z \in \frac{1}{2}\mathbb{N}_0$ the definition is analogous but now the representation is in the space $\{v_{2z}, ..., v_0\}$.

6.2.1 Prior estimate for matrix elements

Let $\lambda_z : \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C}))) \to \mathbb{C}$ be the central character of $\overset{z}{\tilde{\rho}}$. Given a chord diagram w with m chords, we wish to give a rough estimate for $|\lambda_z(\phi_t(w))|$. Consider the inner product $<,>$ on $\overset{z}{V}$ which has an orthonormal basis given by $\{v_{2z}, v_{2z-1}, v_{2z-2}, ...\}$. Obviously $\lambda_z(\phi_t(w)) =< v_{2z}, \phi_t(w)v_{2z} >$. Thus

$$\lambda_z(\phi_t(w)) = \sum_{i=1}^{3^m} \left\langle v_{2z}, A_1^i A_2^i ... A_{2m}^i v_{2z} \right\rangle,$$

where each A_k^i can be $\frac{E}{2}$, $\frac{F}{2}$ or $\frac{H}{2\sqrt{2}}$. The term "3^m" comes from the fact the infinitesimal R-matrix t is a sum of 3 terms. From this formula it is trivial to conclude that $z \to \lambda_z(\phi_t(w))$ is polynomial in z of degree at most $2m$, avoiding a proof using the general theory on central characters and highest weight representations. Also

$$|\lambda_z(\phi_t(w))| \leq \sum_{i=1}^{3^m} \left|\left\langle v_{2z}, A_1^i A_2^i ... A_{2n}^i v_{2z} \right\rangle\right|.$$

Thus from the general expression of the spin z representations it follows that given any $z \in \mathbb{C}$, there exists a $D > 0$ such that for any chord diagram with m chords we have:

$$|\lambda_z(\phi_t(w))| \leq D^m \Gamma(|2z| + m + 1)^2.$$

Applying Lemma 71 we conclude.

Lemma 76 *Let z be a complex number. There exists a constant $C > 0$ such that for any chord diagram with m chords we have:*

$$|\lambda_z(\phi_t(w))| \leq C^m (m!)^2.$$

6.2.2 Refined estimate for matrix elements

The estimate we have just obtained for $\lambda_z(\phi_t(w))$ is not good enough for our purposes, namely to prove Theorem 34. However Theorem 44 in 5.2 (that is theorem 1 of [CV]), permits us to refine this estimate, depending on the kind of chord diagram we have . We freely use the notation of 5.2, and frequently use the normalisation s for the infinitesimal R-matrix of $\mathfrak{sl}(2,\mathbb{C})$. Obviously, Lemma 76 is true if we use ϕ_s instead of ϕ_t. This is going to be a long section with a large amount of notation and nomenclature being introduced.

Definition 77 *Let z be a complex number. Define $c_z = \lambda_z(\phi_s(\ominus))$, here $\ominus$ is the chord diagram with just one chord. It is the action of the rescaled Casimir operator in the representation $\tilde{\overset{z}{\rho}}$ of spin z.*

Definition 78 (Crossing Number) *Let ω be a chord diagram, and a be a chord of it. The crossing number $\times(a,\omega)$ of a in ω is the number of chords of ω that cross the chord a.*

The following definition is the obvious counterpart of Definition 65. The reader is advised to look at it again.

Definition 79 (Normal Chord Diagrams) *Let ω be a chord diagram with m chords. Suppose there are some chords $a^1, ..., a^n$ of ω which do not intersect each*

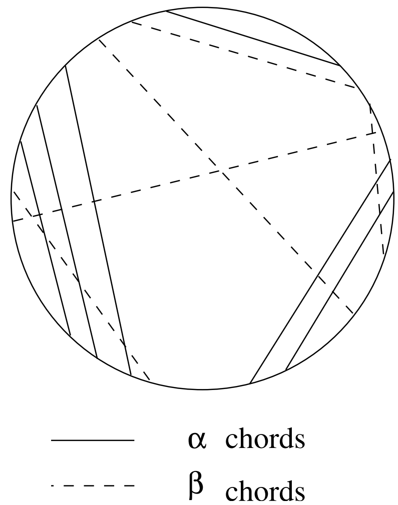

Figure 27: A normal chord diagram showing α-chords and β-chords

others, call them α-chords. For each $m \in 1...n$, let $b_1^m, ..., b_{k_m}^m$ be the chords of ω crossing a^m. Call these chords β_1-chords. Notice that by definition k_m is the crossing number of a^m in ω. We will have in addition a set of chords completing the chord diagram, called β_2-chords. Therefore β_2-chords do not cross α-chords. A β-chord is by definition either a β_1-chord or a β_2-chord. A chord diagram with an indexing of chords like this is called normal. Obviously this definition depends on the way we specify the α, β_1 and β_2-chords, since any α-chord can be called a β-chord. See figure 27.

We have

Lemma 80 *Consider applying Theorem 44 to a normal chord diagram ω and an α-chord $a^m \in \{a^1, .., a^n\}$ of it. All the chord diagram $\omega(a^m)$, $\omega^{||}(a^m, i, j); 1 \leq i < j \leq k_m$ and $\omega^X(a^m, i, j); 1 \leq i < j \leq k_m$ are normal with the same number of β chords, with the set of α-chords losing the chord α^m. Moreover the crossing number*

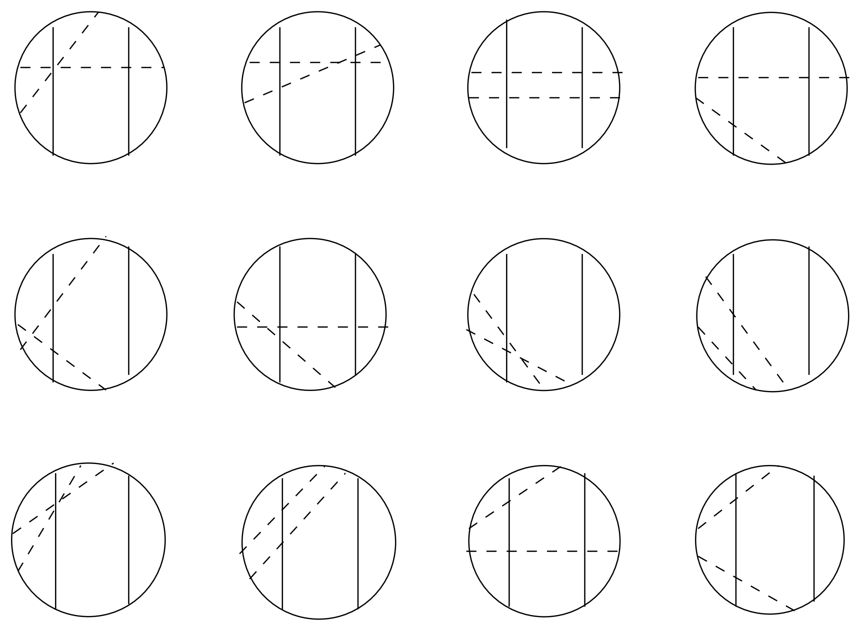

Figure 28: Combinations as in proof of Lemma 80

of each α-chord a^l in each of just defined diagrams is not bigger than k_l. In other words $\times(a^l, \omega(a^m)) \leq \times(a^l, \omega)$ and $\times(a^l, \omega^{||}(a^m, i, j)), \times(a^l, \omega^X(a^m, i, j)) \leq \times(a^l, \omega)$ for $1 \leq i < j \leq k_m$.

Proof. The first assertion is obvious. For the second, let b_i^m and b_j^m cross a^m and let a^l be another α-chord. Obviously the crossing number of a^l in $\omega(a^m)$ equals the crossing number of a^l in ω for the former is obtained from removing a chord that does not cross α^l. The chord diagrams $\omega^{||}(a^m, i, j)$ and $\omega^X(a^m, i, j))$ lose the chords b_i^m and b_j^m, inheriting two new chords. However in both cases, the number of times these two new chords cross a^l is not bigger than the number of times b_i^m and b_j^m crossed a^l. We can see this for example case by case as in figure 28. ∎

Lemma 81 *Let z be a complex number. There exists a $D < +\infty$ such that the* *following is true:*

Let ω be a normal chord diagram with a set $\{a^1, ..., a^n\}$ of α-chords, where a^l has crossing number k_l. Let m be the number of chords of ω and m_β be the number of β chords. We have

$$|(\lambda_z \circ \phi_s)(\omega)| < D^m (m_\beta!)^2 \prod_{l=1}^{n} (2k_l^2 + k_l + |c_z|).$$

Proof. We proceed by induction on m_α, which is the number of α-chords. Choose C as in Lemma 76 and let $D = \max(C, 1)$, thus the assertion is trivial for $m_\alpha = 0$. Let ω be a normal chord diagram with a set $\{\alpha_1, ..., \alpha_n\}$ of α-chords. By Theorem 44 we have

$$\begin{aligned}
&|(\lambda_z \circ \phi_s)(\omega)| = \\
&\quad \left|(\lambda_z \circ \phi_s)\left((\ominus - 2k_n)\omega(a^n) + 2\sum_{1 \leq i < j \leq k_n} \left(\omega^{||}(a^n, i, j) - \omega^X(a^n, i, j)\right)\right)\right| \\
&\quad\quad \leq |(\lambda_z \circ \phi_s)(\ominus)||(\lambda_z \circ \phi_s)(\omega(a^n))| + 2k_n|(\lambda_z \circ \phi_s)(\omega(a^n))| \\
&\quad + 2\sum_{1 \leq i < j \leq k_n} \left(\left|(\lambda_z \circ \phi_s)\left(\omega^{||}(a^n, i, j)\right)\right| + \left|(\lambda_z \circ \phi_s)\left(\omega^X(a^n, i, j)\right)\right|\right) \\
&\quad\quad\quad = |c_z||(\lambda_z \circ \phi_s)(w(a^n))| + \sum_{q=1}^{2k_n^2 + k_n} |(\lambda_z \circ \phi_s)(\omega_q)|,
\end{aligned}$$

where, as consequence of Lemma 80, for any $q \in \{1, ..., k_n^2 + k_n\}$, the diagram ω_q is a normal chord diagram with $m - 1$ chords, having $n - 1$ α-chords and the same number of β-chords. Moreover $\times(a^l, \omega_q) \leq \times(a^l, \omega) = k_l$, for $q = 1, ..., 2k_n^2 +_n$. All this is valid for $w(a)$. By induction, we have:

$$|(\lambda_z \circ \phi_s)(\omega)| \leq D^{m-1}(m_\beta!)^2$$
$$\Big(|c_z| \prod_{l=1}^{n-1} (2(\times^2(a^l, w(a^n)) + \times(a^l, w(a^n)) + |c_z|)$$
$$+ \sum_{q=1}^{2k_n^2+k_n} \prod_{l=1}^{n-1} (2(\times^2(a^l, w_q) + \times(a^l, w_q)) + |c_z|)\Big).$$

Thus

$$\begin{aligned}|(\lambda_z \circ \phi_s)(\omega)| &\leq D^{m-1}(m_\beta!)^2 \left(|c_z| \prod_{l=1}^{n-1} (2k_l^2 + k_l + |c_z|) + \sum_{q=1}^{2k_n^2+k_n} \prod_{l=1}^{n-1} (2k_l^2 + k_l + |c_z|) \right) \\ &\leq D^m (m_\beta!)^2 \prod_{l=1}^{n} (2k_l^2 + k_l + |c_z|).\end{aligned}$$

Where in the last inequality we used the fact $1 \leq D$. ■

Corollary 82 *Let z be a complex number. There exists a $C < +\infty$ such that the following is true:*

Let ω be a normal chord diagram with a set $\{a^1, ..., a^n\}$ of α-chords, where a^l has crossing number k_l. Let m be the number of chords of ω and m_β be the number of β-chords. We have

$$|\lambda_z(\phi_s(\omega))| \leq C^m (m_\beta!)^2 \prod_{l=1}^{n} (k_l + 1)^2.$$

If, moreover, the crossing numbers k_l are all non zero then we can find D such that:

$$|\lambda_z(\phi_s(\omega))| \leq D^m (m_\beta!)^2 \prod_{l=1}^{n} k_l^2.$$

Proof. Choose D like in the just proved lemma, thus:

$$\begin{aligned}
|\lambda_z(\phi_s(\omega))| &\leq D^m (m_\beta!)^2 \prod_{l=1}^{n} (2k_l^2 + k_l + |c_z|) \\
&\leq |D^m| \max(2, |c_z|)^n (m_\beta!)^2 \prod_{l=1}^{n} (k_l^2 + 2k_l + 1) \\
&\leq |D^m| \max(2, |c_z|)^n (m_\beta!)^2 \prod_{l=1}^{n} (k_l + 1)^2 \\
&\leq \max(D, 2, c_z)^{2m} (m_\beta!)^2 \prod_{l=1}^{n} (k_l + 1)^2.
\end{aligned}$$

Notice that $n \leq m$. For the second assertion choose $D = 2C$. ■

6.2.3 The case $\omega = \omega_P$, where P is a pairing

The notation of 6.1.4 is in force now. Let b be a braid with n-strands whose closure is a knot, and let $\underline{m} \in \mathbb{N}_0^{2n+1}$. Any pairing in $\mathcal{P}_{\underline{m}}$ defines a chord diagram ω_P with $|\underline{m}|$ chords. Let z be a complex number, and λ_z the central character of the representation of $\mathfrak{sl}(2, \mathbb{C})$ of spin z. We wish to find an estimate for the norm of $\lambda_z(\phi_s(w_P))$.

Lemma 83 *Let P be a pairing in $\mathcal{P}_{\underline{m}}$ then ω_P is a normal chord diagram where each chord of type A (resp B_1 and B_2) of P generates an α (resp β_1 and β_2) chord of ω_P. In particular all crossing numbers of α-chords in ω_P are non zero since we only consider admissible pairings.*

Proof. This is obvious. The reader is referred to figure 29 ■

The following is trivial as well from figure 29

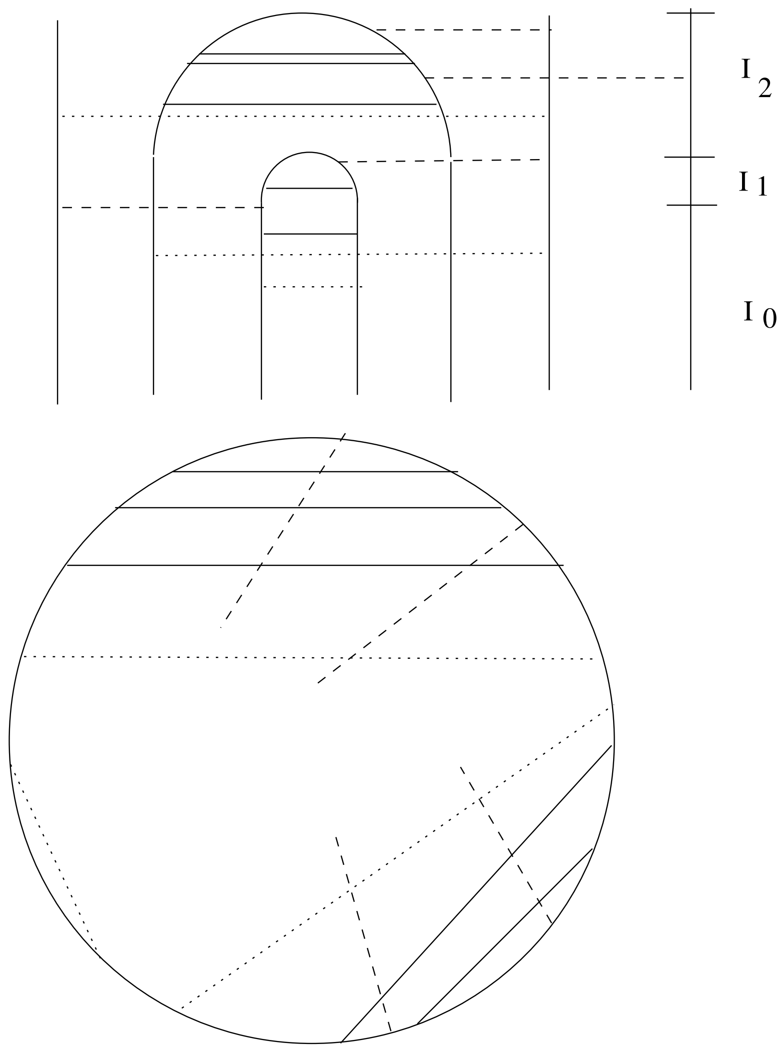

Figure 29: Transforming a pairing into a normal chord diagram

Lemma 84 *Let P be a pairing in $\mathcal{P}_{\underline{m}}$. Consider the type sequences $\mathcal{T}(P^k)$ of $P = (P^{-k}, ..., P^k)$, where $k \in \{-n, ..., n\}$. Each term X in the sequence $\mathcal{T}(P^k)$ generates a chord c_X of ω_P, and if $X = A$ then c_X is an α-chord of ω_P. Suppose $X = A$ and X the i^{th} element of $\mathcal{T}(P^k)$. We have:*

$$\times(c_X, \omega_P) = \#\{j \in \{1, ..., i\} : \mathcal{T}(P^k)(j) = B\}.$$

Thus

Theorem 85 *Let z be a complex number. Consider a braid b with n strands, whose closure is a knot, as well as its closure as defined in 6.1.4. There exists a $C < +\infty$ such that the following is true:*

Let $\underline{m} \in \mathbb{N}_0^{2n+1}$ and $P \in \mathcal{P}_{\underline{m}}$, where $\underline{m} = (m_{-n}, ..., m_n)$, thus P defines a chord diagram w_P with $|\underline{m}|$ chords. We have:

$$|\lambda_z(\phi_s(\omega_P))| < D^{|\underline{m}|}(B(P)!)^2 \prod_{k=-n}^{n} \prod_{\substack{i=\{1,...,m_k\} \\ T(P^k)(i)=A}} \left(\#\{j \in \{1, ..., i\} : \mathcal{T}(P^k)(j) = B\}\right)^2.$$

Proof. This is a direct consequence of Corollary 82 as well lemmas 83 and 84. To apply Corollary 82, notice that $\mathcal{T}(P^k)(1) = B, k = -n, ..., n$. ■

6.2.4 Final ingredients for the proof

Fix once for all a braid b with n strands and consider the closure of it as in 6.1.4. We suppose this closure to be a knot. Recall the map $\psi : \mathcal{A} \to \mathcal{A}$ of 6.1.3, of major importance for the definition of the framed Kontsevich Universal Knot Invariant. We wish to obtain an estimate for the modulus of $(\lambda_z \circ \phi_s)(\psi(w_P))$ where P is a pairing. Recall that the degree $\deg(\omega)$ of a chord diagram ω is given by the number of chords of it. Let ω be a chord diagram, we use the notation $x \subset \omega$ to indicate that x is a chord diagram made out of some of the chords of ω. The notation $\omega \setminus x$ stands for the obvious complementary diagram of x in ω. Recall that

$$\Delta(\omega) = \sum_{x \subset \omega} x \otimes (\omega \setminus x),$$

thus

$$\psi(\omega) = \sum_{x \subset \omega} (-\ominus)^{\deg(x)} (\omega \setminus x).$$

Lemma 86 *Let $z \in \mathbb{C}$ be a complex number and let b be a braid with n strands. There exists a $C < +\infty$ such that for any $\underline{m} = (m_{-n}, ..., m_n) \in \mathbb{N}_0^{2n+1}$ and any pairing $P \in \mathcal{P}_{\underline{m}}$ we have*

$$|\lambda_z(\phi_s(x))| < D^{|\underline{m}|} (B(P)!)^2 \prod_{k=-n}^{n} \prod_{\substack{i=\{1,...,m_k\} \\ \mathcal{T}(P^k)(i)=A}} \left(\#\{j \in \{1, ..., i\} : \mathcal{T}(P^k)(j) = B\}\right)^2, \tag{94}$$

for any subchord diagram $x \subset \omega_P$.

Proof. This is obvious given that the right hand side of the first estimate in Corollary 82 for $(\lambda \circ \phi_s)(\omega)$, where ω is a normal chord diagram, is always reduced if a chord is taken out of it, no matter what its type is. ∎

Let $C = \max(|c_z|, 1)$ (recall $c_z = (\lambda_z \circ \phi_s)(\ominus)$). Consider a chord diagram with m chords ω, and let x be a subchord diagram of it. We have

$$\begin{aligned} \left|(\lambda_z \circ \phi_s)\left((-\ominus)^{\deg(x)}(w \setminus x)\right)\right| &= |c_z|^{\deg(x)} \left|(\lambda_z \circ \phi_s)(w \setminus x)\right| \\ &\leq C^m \left|(\lambda_z \circ \phi_s)(w \setminus x)\right|. \end{aligned}$$

In addition, given a pairing $P \in \mathcal{P}_{\underline{m}}$, there are exactly $2^{|\underline{m}|}$ subchord diagrams of ω_P, since ω_P has $|\underline{m}|$ chords. We can state:

Theorem 87 *Let $z \in C$ be a complex number and b be a braid with n-strands whose closure is a knot. There exists a $C < +\infty$ such that for any $\underline{m} \in \mathbb{N}_0^{2n+1}$ and $P \in \mathcal{P}_{\underline{m}}$ we have:*

$$|(\lambda_z \circ \phi_s)(\psi(\omega_P)))| < C^{|\underline{m}|}(B(P)!)^2 \prod_{k=-n}^{n} \prod_{\substack{i=\{1,...,m_k\} \\ T(P^k)(i)=A}} \left(\#\{j \in \{1,...,i\} : \mathcal{T}(P^k)(j) = B\}\right)^2 .$$

6.2.5 The proof made simple

We now give the actual proof of Theorem 34. We will use the notation of 6.1 freely. Let K be a framed knot, thus it defines an isotopy class of knots which we also call K. We can see K as the closure of some braid b with n strands. Let γ be the Morse parametrisation of K constructed in this way. In particular it has n extreme points.

Recall the Kontsevich Integral is:

$$\begin{aligned}
\mathbf{Z}(\gamma) &= \sum_{m \in \mathbb{N}_0} \frac{1}{(2\pi i)^m} \sum_{\substack{\underline{m} \in \mathbb{N}_0^{2n+1} \\ P \in \mathcal{P}_{\underline{m}}}} (-1)^{\# P} Z(\underline{m}, P) w_P \\
&= \sum_{m \in \mathbb{N}_0} \frac{1}{(2\pi i)^m} \mathbf{Z}(\gamma, m),
\end{aligned}$$

where the convergence is understood in the graded algebra sense. The Unframed Kontsevich Universal Knot Invariant is given by:

$$\mathcal{Z}_u(K) = \frac{\mathbf{Z}(\gamma)}{\mathbf{Z}(\infty)^n} \in \mathcal{A}'.$$

The Framed Kontsevich Universal Knot Invariant expresses as:

$$\mathcal{Z}_f(K) = e^{F(K)\ominus} \psi(\mathcal{Z}_u(K)) \in \mathcal{A}.$$

Since $\psi : \mathcal{A}' \to \mathcal{A}$ is an algebra morphism we can write this as:

$$\mathcal{Z}_f(K) = e^{F(K)\ominus} \frac{\psi(\mathbf{Z}(\gamma))}{\psi(\mathbf{Z}(\infty))^n}.$$

Lemma 88 *Let O be the zero framed unknot. In fact $\mathcal{Z}_f(O) = \psi(\mathbf{Z}(\infty))^{-1}$.*

Proof. Let O_1 be the 1-framed unknot, thus $O_1 = \infty$ as Morse knots. We have:

$$\mathcal{Z}_f(O) = e^{-\ominus} \mathcal{Z}_f(O_1) = e^{-\ominus} e^{F(O_1)\ominus} \frac{\psi(\mathbf{Z}(O_1))}{\psi(\mathbf{Z}(\infty))^2} = \psi(\mathbf{Z}(\infty))^{-1}.$$

■

Observation 89 *Actually we need to know this fact to prove the definition of the Kontsevich Universal Knot Invariant in 6.1.3 is correct. See [LM2], proof of theorem* 5.13 *or [LM3] Proposition* 4.1*.*

Recall that $\phi_{t,h} : \mathcal{A} \to \mathcal{C}(U(\mathfrak{sl}(2,\mathbb{C})))[[h]]$ is the map such that $\omega \to h^{\deg(\omega)}\phi_t(w)$, for any chord diagram ω. Here $\deg(\omega)$ is the number of chords of ω. Let $z \in \mathbb{C}$, and let K be a framed knot. We wish to prove that $\frac{J^z(K)}{2z+1}$ is a power series of Gevrey type 1. Let γ be a Morse parametrisation of K with $2a$ critical points. Since $(\lambda_z \circ \phi_t) : \mathcal{A} \to \mathbb{C}$ is an algebra morphism, we have:

$$\begin{aligned} \frac{J^z}{2z+1}(K) &= (\lambda_z \circ \phi_{t,h})(\mathcal{Z}_f(K)) \\ &= \big((\lambda_z \circ \phi_{t,h})(e^{F(K)\ominus})\big) \left((\lambda_z \circ \phi_{t,h}) Z_f(O)\right)^a \left((\lambda_z \circ \phi_{t,h})(\psi(\mathbf{Z}(\gamma))\right) \\ &= e^{F(K)c_z h} \left(\frac{J^z}{2z+1}(O)\right)^a \left((\lambda_z \circ \phi_{t,h})(\psi(\mathbf{Z}(\gamma)))\right). \end{aligned}$$

Notice that the set of power series $G_1[[h]]$ of Gevrey type 1 is a subalgebra of $\mathbb{C}[[h]]$. Obviously $e^{c_z h} \in G_1[[h]]$. Also

$$\frac{J^z}{2z+1}(O) = \frac{1}{2z+1}\frac{\sinh((2z+1)h/2)}{\sinh(h/2)} \in G_1[[h]].$$

Therefore, the proof of Theorem 34 will be finished once we prove that $(\lambda_z \circ \phi_{t,h})(\psi(\mathbf{Z}(\gamma))$ is of Gevrey type 1. We can suppose γ is a parametrization of the closure of a braid b with n strands, thus $a = n$. From the rescaling property $\phi_t(w) = \left(\frac{-1}{4}\right)^m \phi_s(w)$, for any chord diagram ω with m chords, thus it is equivalent to prove $(\lambda_z \circ \phi_{s,h})(\psi(\mathbf{Z}(\gamma)) \in G_1[[h]]$.

Lemma 90 *Fix a spin $z \in \mathbb{C}$. There exists a $C < \infty$ such that*

$$|(\lambda_z \circ \phi_s)\,(Z(\underline{m}, P)\psi(\omega_P))| < C^{|\underline{m}|}|\underline{m}|!,$$

for any $\underline{m} \in \mathbb{N}_0^{2n+1}$ and pairing $P \in \mathcal{P}_{\underline{m}}$.

Proof. By Theorems 69 and 87, there exists a $D < +\infty$ such that we have:

$$\begin{aligned}
&|(\lambda_z \circ \phi_s)\,(Z(\underline{m}, P)\psi(\omega_P))| \\
&\qquad \le D^{|\underline{m}|} B(P)! \prod_{k=-n}^{n} \prod_{\substack{i=\{1,\ldots,m_k\} \\ T(P^k)(i)=A}} \left(\#\{j \in \{1, ..., i\} : \mathcal{T}(P^k)(j) = B\}\right),
\end{aligned}$$

thus:

$$\begin{aligned}
|(\lambda_z \circ \phi_s)\,(Z(\underline{m}, P)\psi(\omega_P))| &\le D^{|\underline{m}|} B(P)! \prod_{k=-n}^{n} \prod_{\substack{i=\{1,\ldots,m_k\} \\ \mathcal{T}(P^k)(i)=A}} B(P^k) \\
&\le D^{|\underline{m}|} B(P)! \prod_{k=-n}^{n} B(P^k)^{A(P^k)} \\
&= D^{|\underline{m}|} B(P)! B(P)^{A(P)}.
\end{aligned}$$

Recall $A(P)$ equals the number of chords of type A in P, and the same for $B(P)$. By Lemma 72 there exists a $D < +\infty$ such that:

$$|(\lambda_z \circ \phi_s)\left(Z(\underline{m}, P)\psi(\omega_P)\right)| \leq D^{|\underline{m}|} B(P)^{(B(P)+A(P))} = D^{|\underline{m}|} B(P)^{|\underline{m}|} \leq D^{|\underline{m}|} |\underline{m}|^{|\underline{m}|},$$

for any $\underline{m} \in \mathbb{N}_0^{2n+1}$ and $P \in \mathcal{P}_{\underline{m}}$, since for a pairing $P \in \mathcal{P}_{\underline{m}}$ the number of chords $|\underline{m}|$ of P equals $A(P) + B(P)$. Therefore, we just need to use Lemma 72 again. ■

The proof of Theorem 34 is finished if we prove the following lemma.

Lemma 91 *Let b be a braid and γ be a parametrisation of its closure. Fix a spin $z \in \mathbb{C}$. There exists a $C < +\infty$ such that for any $m \in \mathbb{N}_0$ we have:*

$$|(\lambda_z \circ \phi_s)(\psi(\mathbf{Z}(\gamma, m)))| \leq C^m m!.$$

Proof. Obviously

$$|(\lambda_z \circ \phi_s)(\psi(\mathbf{Z}(\gamma, m)))| \leq \sum_{\substack{\underline{m} \in \mathbb{N}_0^{2n+1} \\ |\underline{m}| = m}} \sum_{P \in \mathcal{P}_{\underline{m}}} \left|(\lambda_z \circ \phi_s)(Z(\underline{m}, P)\psi(w_P))\right|,$$

thus, from the previous lemma, the proof reduces to a mere exercise of counting. First of all, the cardinal of $\mathcal{P}_{\underline{m}}$ is at most $[n(2n-1)]^{|\underline{m}|}$, this is because for each interval I_k then $\mathrm{int}(I_k) \cap K$ is made out of at most $2n$ strings, see figure 21. Recall the classical combinatorics problem, which asks about the number of ways we can put X indistinguishable objects into N distinguishable boxes. Its solution is $\frac{(N+X-1)!}{(N-1)!X!}$. In the m^{th} coefficient of the Kontsevich integral we have exactly $X = m$ objects (chords) and $N = 2n+1$ boxes (the intervals of the splitting of $\gamma(S^1)$). In particular there are at most $\frac{(2n+m+1)!}{(2n)!m!}[n(2n-1)]^m$ pairings P with m chords. This last term has exponential growth with respect to m. ■

Therefore

$$(\lambda_z \circ \phi_{s,h})(\psi(\mathbf{Z}(\gamma)) = \sum_{m \in \mathbb{N}} (\lambda_z \circ \phi_s)(\psi(\mathbf{Z}(\gamma, m)))h^m$$

is of Gevrey type 1. This finishes the proof of Theorem 34.

7 The approach with the framework of Buffenoir and Roche

The aim of this chapter is to give a sketch of how the Buffenoir and Roche description of the infinite dimensional unitary representations of the Quantum Lorentz Group as defined by Woronowicz and Podles in [PW] can also be used to define knot invariants. We will see that this approach and the previous perturbative one are equivalent, at least for balanced representations. We will also clarify, I hope, some issues concerning the structure as well as the representation theory of the Quantum Lorentz Group. I would like to acknowledge the excelent work of Buffenoir and Roche, whose influence in the chapter is enormous.

7.1 On the Quantum Lorentz Group

7.1.1 Star structures

Let $\mathcal{A}$ be an Hopf algebra over $\mathbb{C}$ with antipode S. Recall that a star structure is a bijective map $a \in \mathcal{A} \mapsto a^* \in \mathcal{A}$ such that:

1. $(\lambda a)^* = \bar{\lambda} a^*, \forall a \in \mathcal{A}, \forall \lambda \in \mathbb{C}$, the bar denotes complex conjugation,
2. $(a^*)^* = a, \forall a \in \mathcal{A}$,
3. $(ab)^* = b^* a^*, \forall a, b \in \mathcal{A}$,
4. $\Delta \circ * = (* \otimes *)\Delta$.

Since the antipode in any Hopf algebra is unique, if follows that S is invertible and that:

$$S \circ * = * \circ S^{-1}, \text{ that is } * \circ S = S^{-1} \circ *. \tag{95}$$

A Hopf algebra provided with a star structure will be called a star Hopf algebra. A representation ρ of a star Hopf algebra $\mathcal{A}$ in a pre Hilbertean space V is called unitary if $\rho(a)^\dagger = \rho(a^*), \forall a \in \mathcal{A}$. In other words if:

$$\langle \rho(a)v, w \rangle =< v, \rho(a^*)w >, \forall v, w \in V, \forall a \in \mathcal{A}.$$

Let a be an hermitean element of $\mathcal{A}$, that is an element such that $a^* = a$. Therefore $\rho(a) : V \to V$ is an hermitic operator. We are in a purely algebraic setting now, so we do not impose that the operator $\rho(a)$ admit a unique selfadjoint closure, in other words that it is an essentially self adjoint operators in the Hilbert completion of V. The unitary representations of the Quantum Lorentz Group which we are going to consider later satisfy the former weaker condition of unitarity. It would be interesting to analyse if they also satisfy this stronger condition of topological unitarity. This is something that happens if we consider the derived representations of a unitary representation of a Lie group. See [Kir] or 3.1. In other words we have:

Theorem 92 *Let G be a real Lie group. Let also $\mathfrak{g}$ be the Lie algebra of it and $\mathfrak{g}_{\mathbb{C}}$ be the complexification of its universal enveloping algebra. Therefore there exists a star structure in $\mathfrak{g}_{\mathbb{C}}$ selecting $\mathfrak{g}$ as its eigenspace associated with -1. Let R be a unitary representation of $\mathfrak{g}$.*

1. *The derived representation R^∞ of $\mathfrak{g}_{\mathbb{C}}$ in V_∞ is unitary*

2. *For any $X \in \mathfrak{g}$ the operator $R^\infty(iX) : V_\infty \to V_\infty$ is essentially self adjoint.*

Part 1 of this theorem justifies looking at unitary representations of the quantum Lorentz group as being quantised counterparts of unitary representations of the Lorentz Group. In the quantised version of the thery it is also natural to consider the following compactibility restriction:

Definition 93 *Let $\mathcal{A}$ be a star Hopf algebra with an R-matrix $R \in \mathcal{A} \otimes \mathcal{A}$. We say that R is compactible*[10] *with the star structure if $R^{*\otimes *} = R^{-1}$.*

One reason to consider this compatibility relation is:

Proposition 94 *Let ρ be a unitary representation of $\mathcal{A}$ in the pre hilbertean space V. If R is compactible with $*$ then the braiding operators:*

$$B_{V,V} : v \otimes w \in V \otimes V \mapsto R_{21}(w \otimes v) \in V \otimes V$$

are unitary, in other words:

$$\langle B_{V,V}(v \otimes w), B_{V,W}(v' \otimes w') \rangle = \langle v \otimes w, v' \otimes w' \rangle \,.$$

Therefore they uniquely extend to a topologically unitary map $\overline{V \otimes V} \to \overline{V \otimes V}$ where $\overline{V \otimes V}$ is the Hilbert completion of $V \otimes V$.

7.1.2 The algebra $U_q(\mathfrak{su}(2))$ and Clebsch-Gordan Coefficients

Let $q \in (0, 1)$. The reason why we make this restriction concernes the fact that the star structure whihch we are going to consider in $U_q(\mathfrak{su}(2))$ only makes sense if q is real and not $0, 1, -1$, so we are merely taking a connected component of this set,

[10]Majid call these quasitriangular structures anti-real as oposed to real ones for which $R^{*,*} = R_{21}$. For example if q is real, the usual R-matrix of $U_q(\mathfrak{su}(2))$ is real. See [M] and bellow

probably an artificial restriction. Consider the Hopf algebra $U_q(\mathfrak{su}(2))$ as defined for example in [BR2]. It is the unital star Hopf algebra defined by the generators J_+, J_-, q^{J_z} and q^{-J_z} with relations:

$$q^{\pm J_z} q^{\mp J_z} = 1, \quad q^{J_z} J_\pm q^{-J_z} = q^{\pm 1} J_\pm, \quad [J_+, J_-] = \frac{q^{2J_z} - q^{-2J_z}}{q - q^{-1}}, \tag{96}$$

$$\Delta(q^{\pm J_z}) = q^{\pm J_z} \otimes q^{\pm J_z}, \quad \Delta(J_\pm) = q^{-J_z} \otimes J_\pm + J_\pm \otimes q^{J_z}, \tag{97}$$

$$S(q^{\pm J_z}) = q^{\mp J_z}, \quad S(q^{J_\pm}) = -q^{\pm 1} J_\mp, \tag{98}$$

$$\varepsilon(q^{\pm J_z}) = 1, \quad \varepsilon(J_\pm) = 0, \tag{99}$$

$$(q^{\pm J_z})^* = q^{\pm J_z}, \quad J_\pm^* = q^{\mp 1} J_\mp. \tag{100}$$

Notice it is a slightly unstandard definition, but this will not affect our discussion. This algebra is denoted by $\hat{U}_q(\mathfrak{su}(2)$ in [KS]. It is a semisimple algebra, that is any finite dimensional representation of it can be expressed uniquely as a direct sum of irreducibles. The irreducible finite dimensional representations of $U_q(\mathfrak{su}(2))$ are classified by a $w \in \{1, -1, i, -i\}$ and an $\alpha \in \frac{1}{2}\mathbb{N}_0$. We denote these irreducible representations by $\overset{\alpha}{\rho}_w$. The representations $\overset{\alpha}{\rho}$ of spin $\alpha \in \frac{1}{2}\mathbb{N}_0$ are, by definition, the ones for which $w = 1$. They are the natural quantisation of the representations of $\mathfrak{su}(2)$ of spin α. The representation space for the representation $\overset{\alpha}{\rho}_w$ is $\overset{\alpha}{V}_w$, a complex vector space with dimension $2\alpha+1$. There exists a basis $\{\overset{\alpha}{v}_{-\alpha}, ..., \overset{\alpha}{v}_a\}$ of $\overset{\alpha}{V}_w$ on which the expression of the representations $\overset{\alpha}{\rho}_w$ is:

$$q^{\pm J_z} \overset{\alpha}{v}_{i_\alpha} = w q^{\pm i_\alpha} \overset{\alpha}{v}_{i_\alpha} \tag{101}$$

$$J_+ \overset{\alpha}{v}_{i_\alpha} = w q^{-\frac{1}{2}} \sqrt{[\alpha + i_\alpha + 1][\alpha - i_\alpha]} \overset{\alpha}{v}_{i_\alpha + 1} \tag{102}$$

$$J_- \overset{\alpha}{v}_{i_\alpha} = w q^{+\frac{1}{2}} \sqrt{[\alpha - i_\alpha + 1][\alpha + i_\alpha]} \overset{\alpha}{v}_{i_\alpha - 1} \tag{103}$$

Where the quantum integers $[n]_q = [n]$ are defined as $[n] = \frac{q^{-n}-q^n}{q^{-1}-q}$. Notice that the four unique irreducible representations of dimension $2\alpha+1$ of $U_q(\mathfrak{su}(2))$ are distinguished by the spectrum of q^{J_z}. This fact will be useful later on.

Let $\overset{\alpha}{V} = \overset{\alpha}{V}_w$ be the representation space of the representation of spin α. The representations $\overset{\alpha}{\rho}$ are unitary under an inner product in $\overset{\alpha}{V}$ such that $\{\overset{\alpha}{v}_{-\alpha}, ..., \overset{\alpha}{v}_a\}$ is an orthonormal basis. We need to assume $q \in (0,1)$ for this to hold, and actually for the star structure in $U_q(\mathfrak{su}(2))$ to be well defined in the first place. The quantum dimension of the representation $\overset{\alpha}{\rho}$ is $d_\alpha = [2\alpha+1]_q$. The representation of spin zero is the representation on $\mathbb{C}$ such that $a\lambda = \varepsilon(a)\lambda, a \in U_q(\mathfrak{su}(2))\lambda \in \mathbb{C}$.

Similarly to the classical case, the representations $\overset{\alpha}{V}$ are closed under taking tensor products and direct sums in the sense:

$$\overset{\alpha}{V} \otimes \overset{\beta}{V} \cong \bigoplus_{\gamma=|\alpha-\beta|}^{\alpha+\beta} \overset{\gamma}{V}. \tag{104}$$

There exist thus intertwiners $\Psi_\gamma^{\alpha,\beta} : \overset{\alpha}{V} \otimes \overset{\beta}{V} \to \overset{\gamma}{V}$ and $\Phi_{\alpha,\beta}^\gamma : \overset{\gamma}{V} \to \overset{\alpha}{V} \otimes \overset{\beta}{V}$ called the Clebsch-Gordan maps. They are defined apart from phases, but after making some natural choices there is a well determined one. All this is described in [BR2] and [KS]. We define the Clebsch-Gordan coefficients as:

$$\Psi_\gamma^{\alpha,\beta}(\overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta}) = \sum_{i_\gamma} \left(\begin{array}{c|cc} i_\gamma & \alpha & \beta \\ \gamma & i_\alpha & i_\beta \end{array}\right) \overset{\gamma}{v}_{i_\gamma}, \tag{105}$$

and

$$\Phi_{\alpha,\beta}^\gamma(\overset{\gamma}{v}_{i_\gamma}) = \sum_{i_a, i_b} \left(\begin{array}{cc|c} i_\alpha & i_\beta & \gamma \\ \alpha & \beta & i_\gamma \end{array}\right) \overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta}. \tag{106}$$

Under the phases chosen in [BR2] they are always real. Let $Y(\alpha,\beta,\gamma) = 1$ if $\overset{\alpha}{V}$ is in the decomposition of $\overset{\beta}{V} \otimes \overset{\gamma}{V}$ as a direct sums of irreducibles, and zero otherwise.

Observe that $Y(\alpha, b, \gamma)$ is symmetric in its arguments. If $Y(\alpha, \beta, \gamma) \neq 0$ we say the triple α, β, γ is admissible. Let also $Y(\alpha, i_\alpha) = 1$ if $i_\alpha \in \{-\alpha, ..., a\}$, and zero otherwise. The Clebsch-Gordan coefficients admit the following explicit formula.

$$\left(\begin{matrix} m & n \\ I & J \end{matrix} \middle| \begin{matrix} K \\ p \end{matrix}\right) = Y(I, m)Y(J, n)Y(K, p)\delta(m + n, p)Y(I, J, K)$$
$$q^{m(p+1)+\frac{1}{2}((J(J+1)-I(I+1)-K(K+1)}e^{i\pi(I-m)}$$
$$\sqrt{\frac{[2K+1][I+J-K]![I-m]![J-n]![K-p]![K+p]!}{[K+J-I]![I+K-J]![I+J+K+1]![I+m]![J+n]!}}$$
$$\times \sum_{\substack{V=0 \\ -J+K-m \leq V \leq I-m}}^{K-p} \frac{q^{V(K+p+1)}e^{i\pi V}[I+m+V]![J+K-m-V]!}{[V]![K-p-V]![I-m-V]![J-K+m+V]!} \tag{107}$$

which will be useful later on. See [BR2] or [KS]. Observe that $-J + K - m \leq V \leq I - m$ is equivalent to $I + m + V \geq 0$, $J - K + m + V \geq 0$, $J + K - m - V \geq 0$ and $I - m - V \geq 0$, under the admissibility rules.

The Clebsch-Gordan Coefficients satisfy the following well known properties (see [BR2] or [KS]):

$$\left(\begin{matrix} i_\gamma \\ \gamma \end{matrix} \middle| \begin{matrix} 0 & \beta \\ 0 & i_\beta \end{matrix}\right) = \delta(\gamma, \beta)\delta(i_\gamma, i_\beta), \tag{108}$$

$$\left(\begin{matrix} i_\gamma \\ \gamma \end{matrix} \middle| \begin{matrix} \alpha & 0 \\ i_\alpha & 0 \end{matrix}\right) = \delta(\gamma, \alpha)\delta(i_\gamma, i_\alpha). \tag{109}$$

In the following two equations the implications are not equivalences: in the quantum case a Clebsch-Gordan Coefficient can be zero when you do not expect it to be zero, for some special values of q. These are known as special zeros of Clebsh-Gordon

Coefficients.

$$\begin{pmatrix} i_\gamma & \Big| & \alpha & \beta \\ \gamma & \Big| & i_\alpha & i_\beta \end{pmatrix} \neq 0 \implies i_\gamma = i_\alpha + i_\beta, \tag{110}$$

$$\begin{pmatrix} i_\alpha & i_\beta & \Big| & \gamma \\ \alpha & \beta & \Big| & i_\gamma \end{pmatrix} \neq 0 \implies i_\gamma = i_\alpha + i_\beta. \tag{111}$$

We also have:

$$\begin{pmatrix} i_\gamma & \Big| & \alpha & \beta \\ \gamma & \Big| & i_\alpha & i_\beta \end{pmatrix} = \begin{pmatrix} i_\alpha & i_\beta & \Big| & \gamma \\ \alpha & \beta & \Big| & i_\gamma \end{pmatrix}, \tag{112}$$

this last condition is used to fix the phases of the Clebsch-Gordan coefficients in [BR2]. In addition:

$$\begin{pmatrix} i_\gamma & \Big| & \alpha & \beta \\ \gamma & \Big| & i_\alpha & i_\beta \end{pmatrix} = \begin{pmatrix} -i_\gamma & \Big| & \beta & \alpha \\ \gamma & \Big| & -i_\beta & -i_\alpha \end{pmatrix}, \tag{113}$$

$$q^{-i_\alpha}(-1)^{i_\alpha} \begin{pmatrix} i_\gamma & \Big| & \alpha & \beta \\ \gamma & \Big| & i_\alpha & i_\beta \end{pmatrix} = e^{i\pi(\beta-\gamma)} \left(\frac{d_\gamma}{d_\beta}\right)^{\frac{1}{2}} \begin{pmatrix} -i_\alpha & i_\gamma & \Big| & \beta \\ \alpha & \gamma & \Big| & i_\beta \end{pmatrix}. \tag{114}$$

In this equation, notice that $\beta - \gamma - i_\alpha$ is always an integer. Similarly with the following formula:

$$q^{i_\beta}(-1)^{-i_\beta} \begin{pmatrix} i_\gamma & \Big| & \alpha & \beta \\ \gamma & \Big| & i_\alpha & i_\beta \end{pmatrix} = e^{i\pi(\alpha-\gamma)} \left(\frac{d_\gamma}{d_\alpha}\right)^{\frac{1}{2}} \begin{pmatrix} i_\gamma & -i_\beta & \Big| & \alpha \\ \gamma & \beta & \Big| & i_\alpha \end{pmatrix}. \tag{115}$$

These are the obvious orthogonality relations:

$$\sum_{i_\alpha + i_\beta = i_\gamma} \begin{pmatrix} i_\gamma & \Big| & \alpha & \beta \\ \gamma & \Big| & i_\alpha & i_\beta \end{pmatrix} \begin{pmatrix} i_\alpha & i_\beta & \Big| & \delta \\ \alpha & \beta & \Big| & i_\delta \end{pmatrix} = \delta(\gamma,\delta)\delta(i_\gamma, i_\delta)Y(\alpha,\beta,\gamma), \tag{116}$$

and

$$\sum_{\gamma, i_\gamma} \begin{pmatrix} i_\alpha & i_\beta & \Big| & \gamma \\ \alpha & \beta & \Big| & i_\gamma \end{pmatrix} \begin{pmatrix} i_\gamma & \Big| & \alpha & \beta \\ \gamma & \Big| & j_\alpha & j_\beta \end{pmatrix} = Y(\alpha, i_\alpha)Y(\beta, i_\beta)\delta(i_a, j_\alpha)\delta(i_\beta, j_\beta). \tag{117}$$

Observe that since the Clebsch-Gordan coefficients are real, (116) together with (112) ensure that their norm is smaller than 1. In some particular cases, the expression for Clebsch-Gordan coefficients simplifies dramatically. For example:

$$\begin{pmatrix} 0 & \Bigg| & A & B \\ 0 & & m & -m \end{pmatrix} = \frac{e^{i\pi(B-m)}}{\sqrt{d_B}} \delta(A,B) q^m. \tag{118}$$

7.1.3 6j-symbols and their symmetries

Again we follow the conventions of [BR2]. All the material here can be easily found in the literature, so we stick to the minimum essential. See [KS] for proofs. Consider the space of $U_q(\mathfrak{su}(2))$ intertwiners $\overset{\alpha}{V} \otimes \overset{\beta}{V} \to \overset{\gamma}{V} \otimes \overset{\beta}{V}$ its has two basis, namely

$$\left\{ \phi^{\varepsilon}_{\gamma,\delta} \psi^{\alpha,\beta}_{\varepsilon}, Y(\alpha,\beta,\varepsilon), Y(\gamma,\delta,\varepsilon) = 1 \right\},$$

and

$$\left\{ \left(\mathrm{id}_{\underset{V}{\gamma}} \otimes \psi^{\omega\beta}_{\delta} \right) \left(\phi^{\alpha}_{\gamma\omega} \otimes \mathrm{id}_{\underset{V}{\beta}} \right), Y(\alpha,\gamma,\omega), Y(\omega,\beta,\delta) = 1 \right\}.$$

The 6j-symbols are defined as

$$\sum_{\varepsilon} \begin{Bmatrix} \gamma & \omega & \Bigg| & \alpha \\ \beta & \varepsilon & & \delta \end{Bmatrix} \phi^{\varepsilon}_{\gamma,\delta} \psi^{\alpha,\beta}_{\varepsilon} = \left(\mathrm{id}_{\underset{V}{\gamma}} \otimes \psi^{\omega\ b}_{\delta} \right) \left(\phi^{\alpha}_{\gamma\omega} \otimes \mathrm{id}_{\underset{V}{\beta}} \right), \tag{119}$$

with the convention

$$\begin{Bmatrix} \gamma & \omega & \Bigg| & \alpha \\ \beta & \varepsilon & & \delta \end{Bmatrix} = 0 \text{ if } Y(\varepsilon,\gamma,\beta) = 0 \text{ or } Y(\varepsilon,\alpha,b) = 0.$$

Therefore

$$\begin{Bmatrix} A & B & \Bigg| & C \\ D & E & & F \end{Bmatrix} = Y(A,B,C)Y(C,D,E)Y(A,E,F)Y(D,B,F) \begin{Bmatrix} A & B & \Bigg| & C \\ D & E & & F \end{Bmatrix}. \tag{120}$$

Obviously we have:

$$\sum_{U,u} \left\{ \begin{array}{cc|c} A & E & C \\ B & U & D \end{array} \right\} \left(\begin{array}{cc|c} a & d & U \\ A & D & u \end{array} \right) \left(\begin{array}{c|cc} u & C & B \\ U & c & b \end{array} \right) = \sum_{e} \left(\begin{array}{c|cc} d & E & B \\ D & e & b \end{array} \right) \left(\begin{array}{cc|c} a & e & C \\ A & E & c \end{array} \right), \quad (121)$$

which by (112) translates into:

$$\sum_{U,u} \left\{ \begin{array}{cc|c} A & E & C \\ B & U & D \end{array} \right\} \left(\begin{array}{cc|c} c & b & U \\ C & B & u \end{array} \right) \left(\begin{array}{c|cc} u & A & D \\ U & a & d \end{array} \right) = \sum_{e} \left(\begin{array}{c|cc} c & A & E \\ C & a & e \end{array} \right) \left(\begin{array}{cc|c} e & b & D \\ E & B & d \end{array} \right). \quad (122)$$

From this and (114) we deduce:

$$q^{c-e}(-1)^{e-c} \sum_{D,d} \frac{e^{i\pi(B+A-2D)}[d_D]}{d_B^{\frac{1}{2}} d_A^{\frac{1}{2}}} \left\{ \begin{array}{cc|c} C & D & B \\ E & M & A \end{array} \right\} \left(\begin{array}{c|cc} a & D & C \\ A & d & c \end{array} \right) \left(\begin{array}{cc|c} e & d & B \\ E & D & b \end{array} \right) = \sum_{m} \left(\begin{array}{c|cc} a & E & M \\ A & -e & m \end{array} \right) \left(\begin{array}{cc|c} m & -c & B \\ M & C & b \end{array} \right). \quad (123)$$

Explicit expressions for $6j$-symbols appear in [BR2] [KS], for example. We have: (these are classical results):

$$\left\{ \begin{array}{cc|c} A & B & E \\ C & D & F \end{array} \right\} = \left\{ \begin{array}{cc|c} B & A & E \\ D & C & F \end{array} \right\} = \left\{ \begin{array}{cc|c} C & D & E \\ A & B & F \end{array} \right\} = \left\{ \begin{array}{cc|c} A & D & F \\ C & B & E \end{array} \right\}, \quad (124)$$

$$\sum_{C} \left\{ \begin{array}{cc|c} A & B & C \\ G & H & I \end{array} \right\} \left\{ \begin{array}{cc|c} A & B & C \\ G & H & J \end{array} \right\} = \delta(I,J)Y(A,H,I)Y(B,G,I). \quad (125)$$

This last relation is known as orthogonality relation. The $6j$-symbols also satisfy the pentagonal equation:

$$\sum_A \left\{\begin{matrix} D & F \\ I & G \end{matrix}\;\middle|\;\begin{matrix} A \\ J \end{matrix}\right\}\left\{\begin{matrix} D & F \\ E & B \end{matrix}\;\middle|\;\begin{matrix} A \\ C \end{matrix}\right\}\left\{\begin{matrix} E & A \\ G & H \end{matrix}\;\middle|\;\begin{matrix} B \\ I \end{matrix}\right\}$$
$$= \left\{\begin{matrix} E & F \\ J & H \end{matrix}\;\middle|\;\begin{matrix} C \\ I \end{matrix}\right\}\left\{\begin{matrix} D & C \\ H & G \end{matrix}\;\middle|\;\begin{matrix} B \\ J \end{matrix}\right\}, \quad (126)$$

known as Biedenharn-Elliott identity.

7.1.4 The algebra $\mathrm{Pol}(SU_q(2))$

This section and the next are to be read simultaneously. Let $\alpha \in \frac{1}{2}\mathbb{N}_0$. Consider the representation $\overset{\alpha}{\rho}_1 = \overset{\alpha}{\rho}$. Its representation space $\overset{\alpha}{V}$ has a basis $\{\overset{\alpha}{v}_{-\alpha}, ..., \overset{\alpha}{v}_{\alpha}\}$. Consider its dual basis $\{\overset{\alpha}{v}^{-\alpha}, ..., \overset{\alpha}{v}^{a}\}$ of $\overset{\alpha *}{V}$. We define the matrix elements:

$$\overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha} = \left\langle \overset{\alpha}{v}{}^{i_\alpha} | \overset{\alpha}{\rho} | \overset{\alpha}{v}_{i_\beta} \right\rangle,$$

where $\alpha \in \frac{1}{2}\mathbb{N}_0, -\alpha \leq i_\alpha, j_\alpha \leq \alpha$. Notice the matrix elements belong to the dual of $U_q(\mathfrak{su}(2))$. It is possible to prove that they are linearly independent, see [KS]. Actually, they generate an Hopf algebra $\mathrm{Pol}(SU_q(2)) \subset U_q(\mathfrak{su}(2))^*$. Let us describe its structure explicitely: The coproduct in $\mathrm{Pol}(SU_q(2))$ is defined by duality as:

$$\begin{aligned} \Delta(\overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha}) = \sum_{k=-\alpha}^{\alpha} \overset{\alpha}{g}{}^{i_\alpha}_{k} \otimes \overset{\alpha}{g}{}^{k}_{j_\alpha} &\in \mathrm{Pol}(SU_q(2)) \otimes \mathrm{Pol}(SU_q(2)) \\ &\subset U_q(\mathfrak{su}(2))^* \otimes U_q(\mathfrak{su}(2))^* \\ &\subset (U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2)))^*, \end{aligned}$$

that is:

$$\Delta(\overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha})(a \otimes b) = \overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha}(ab), \forall a, b \in U_q(\mathfrak{su}(2)).$$

The product in $\mathrm{Pol}(SU_q(2))$ is the convolution product in $U_q(\mathfrak{su}(2))^*$, in other words:

$$\overset{\alpha i_\alpha}{g}_{j_\alpha}\overset{\beta i_\beta}{g}_{j_\beta} = \sum_{\gamma,i_\gamma,j_\gamma} \left(\begin{array}{cc|c} i_\alpha & i_\beta & \gamma \\ \alpha & \beta & i_\gamma \end{array}\right)\left(\begin{array}{c|cc} j_\gamma & \alpha & \beta \\ \gamma & j_\alpha & j_\beta \end{array}\right)\overset{\gamma i_\gamma}{g}_{j_\gamma}. \tag{127}$$

This means:

$$\left(\overset{\alpha i_\alpha}{g}_{j_\alpha}\overset{\beta i_\beta}{g}_{j_\beta}\right)(a) = \left(\overset{\alpha i_\alpha}{g}_{j_\alpha}\otimes\overset{\beta i_\beta}{g}_{j_\beta}\right)(\Delta(a)) = \left\langle \overset{\alpha i_\alpha}{v}\otimes\overset{\beta i_\beta}{v}|(\overset{\alpha}{\rho}\otimes\overset{\beta}{\rho})(a)|\overset{\alpha}{v}_{j_\alpha}\otimes\overset{\beta}{v}_{j_\beta}\right\rangle, \forall a\in U_q(\mathfrak{su}(2)).$$

The identity for the product is $\overset{00}{g}_0$. Notice $\overset{00}{g}_0(a)=\varepsilon(a), \forall a\in U_q(\mathfrak{su}(2))$, by definition of the representation of spin 0 Also $\varepsilon\left(\overset{\alpha i_\alpha}{g}_{j_\alpha}\right)=\overset{\alpha i_\alpha}{g}_{j_\alpha}(1)=\delta(i_\alpha,j_\alpha)$.

The antipode is defined as $S(\overset{\alpha i_\alpha}{g}_{j_\alpha})(a)=\overset{\alpha i_\alpha}{g}_{j_\alpha}(S(a)), \forall a\in U_q(\mathfrak{su}(2))$. It is not immediate that $S(\overset{\alpha i_\alpha}{g}_{j_\alpha})\in \mathrm{Pol}(SU_q(2))$. Let us explain why it is so: Let $\alpha\in\frac{1}{2}\mathbb{N}_0$, and consider the representation $\overset{\alpha}{\rho}$ in $\overset{\alpha}{V}$. Its conjugate representation $\overline{(\overset{\alpha}{\rho})}$ is the representation in the conjugate vector space $\overline{\overset{\alpha}{V}}$ given by $\overline{\overset{\alpha}{\rho}}(a)=\overline{\overset{\alpha}{\rho}(S^{-1}(a^*))}, \forall a\in U_q(\mathfrak{su}(2))$. The conjugate vector space $\overline{\overset{\alpha}{V}}$ admits the basis $\{\overline{\overset{\alpha}{v}_{-\alpha}},...,\overline{\overset{\alpha}{v}_{\alpha}}\}$. An easy calculation shows that $\overset{\alpha}{\rho}$ is equivalent to $\overline{\overset{\alpha}{\rho}}$ and that an intertwiner is: $\overset{\alpha}{v}_{i_\alpha}\mapsto q^{-i_\alpha}(-1)^{\alpha-i_\alpha}\overline{\overset{\alpha}{v}_{-i_\alpha}}$. See [BR2]. We thus have:

$$\left\langle\overset{\alpha i_\alpha}{v}|\overset{\alpha}{\rho}(S(a))|\overset{\alpha}{v}_{j_\alpha}\right\rangle = \overline{\left\langle\overset{\alpha j_\alpha}{v}|\overset{\alpha}{\rho}(S(a)^*)|\overset{\alpha}{v}_{i_\alpha}\right\rangle}, \qquad \text{since } \overset{\alpha}{\rho} \text{ is unitary} \tag{128}$$

$$= \overline{\left\langle\overset{\alpha j_\alpha}{v}|\overset{\alpha}{\rho}(S^{-1}(a^*))|\overset{\alpha}{v}_{i_\alpha}\right\rangle}, \qquad \text{from equation (95)} \tag{129}$$

$$= \left\langle\overline{\overset{\alpha j_\alpha}{v}}|\overline{\overset{\alpha}{\rho}}(a)|\overline{\overset{\alpha}{v}_{i_\alpha}}\right\rangle, \qquad \text{by definition} \tag{130}$$

$$= \left\langle\overset{\alpha -j_\alpha}{v}|\overset{\alpha}{\rho}(a)|\overset{\alpha}{v}_{-i_\alpha}\right\rangle q^{i_\alpha-j_\alpha}(-1)^{-i_\alpha+j_\alpha}. \tag{131}$$

In particular

$$S(\overset{\alpha i_\alpha}{g}_{j_\alpha}) = q^{i_\alpha-j_\alpha}(-1)^{-i_\alpha+j_\alpha}\overset{\alpha -j_\alpha}{g}_{-i_\alpha}\in\mathrm{Pol}(SU_q(2)), \tag{132}$$

if $\alpha \in \frac{1}{2}\mathbb{N}_0$ and $i_\alpha, j_\alpha \in \{-\alpha, ..., \alpha\}$. Observe that S is indeed an antipode for $\mathrm{Pol}(SU_q(2))$, since for any $a \in U_q(\mathfrak{su}(2))$ we have:

$$\begin{aligned}\sum_{(\overset{\alpha i_\alpha}{g}_{j_\alpha})} S(\overset{\alpha i_\alpha}{g}_{j_\alpha}{}')\overset{\alpha i_\alpha}{g}_{j_\alpha}{}''(a) &= \sum_{(a)} \left(S(\overset{\alpha i_\alpha}{g}_{j_\alpha}{}')\overset{\alpha i_\alpha}{g}_{j_\alpha}{}''\right)(a' \otimes a'') \\ &= \overset{\alpha i_\alpha}{g}_{j_\alpha}(\sum_{(a)} S(a')a'') \\ &= \varepsilon(a)\overset{\alpha i_\alpha}{g}_{j_\alpha}(1) \\ &= \overset{00}{g}_{0}(a)\varepsilon(\overset{\alpha i_\alpha}{g}_{j_\alpha}),\end{aligned}$$

where if $\mathcal{A}$ is a Hopf algebra we denote $\Delta(a) = \sum_{(a)} a' \otimes a''$. In particular, given that the matrix elements $\overset{\alpha i_\alpha}{g}_{j_\alpha} \in U_q(\mathfrak{su}(2))^*$ are linearly independent we have:

$$\sum_k S(\overset{\alpha i_\alpha}{g}_{k})\overset{\alpha k}{g}_{j_\alpha} = \delta(i_\alpha, j_\alpha)\overset{00}{g}_{0},$$

which implies the following identity satisfied by Clebsch-Gordan coefficients:

$$\sum_{r,s,k} \left(\begin{array}{cc|c} -k & k & \gamma \\ \alpha & \alpha & r \end{array}\right)\left(\begin{array}{c|cc} s & \alpha & \alpha \\ \gamma & -i_\alpha & j_\alpha \end{array}\right) q^{i_\alpha - k}(-1)^{-i_\alpha + k} = \delta(\gamma, 0)\delta(i_\alpha, j_\alpha). \quad (133)$$

It is possible to give a direct proof of it using the properties of the Clebsch-Gordan coeefficients given before, namely (116) and (118).

The algebra $\mathrm{Pol}(SU_q(2))$ also has a star structure given by $f^*(a) = \overline{f(S^{-1}(a^*)}$. Thus

$$\overset{\alpha i_\alpha}{g}_{j_\alpha}{}^* = S(\overset{\alpha j_\alpha}{g}_{i_\alpha}) = q^{j_\alpha - i_\alpha}(-1)^{i_\alpha - j_\alpha}\overset{\alpha - i_\alpha}{g}_{-j_\alpha}, \quad (134)$$

this follows from (128) and following calculations.

Usually the algebra $\mathrm{Pol}(SU_q(2))$ is denoted by $SU_q(2)$, and can be described in terms of the generators $\{a, b, c, d\}$ and relations as for example in [K], [KS] and also

[BR2]. In fact as an algebra $\mathrm{Pol}(SU_q(2))$ can be defined has the unital algebra with generators $\{a, b, c, d\}$ and relations:

$$qab = ba \quad qac = ca \quad qbd = db \quad bc = ad \quad ad - da = (q^{-1} - q)bc \quad ad - q^{-1}bc = 1. \tag{135}$$

The exact relation between a, b, c, d and the matrix elements of $\mathrm{Pol}(SU_q(2))$ is

$$a = \overset{\frac{1}{2}-\frac{1}{2}}{g}_{-\frac{1}{2}}, \quad b = \overset{\frac{1}{2}-\frac{1}{2}}{g}_{+\frac{1}{2}}, \quad c = \overset{\frac{1}{2}+\frac{1}{2}}{g}_{-\frac{1}{2}}, d = \overset{\frac{1}{2}+\frac{1}{2}}{g}_{+\frac{1}{2}}. \tag{136}$$

See [BR2].We have:

Theorem 95 *The elements a, b, c and d generate* $\mathrm{Pol}(SU_q(2))$ *as an algebra.*

This is because the representation of spin 1/2 generates the representation ring of $U_q(\mathfrak{su}(2))$.

It is well know that the assignment $a \mapsto -a$, $b \mapsto -b$, $c \mapsto -c$ and $d \mapsto -d$ defines a unique automorphism τ of $\mathrm{Pol}(SU_q(2))$ as an algebra. Notice that τ does not preserve the coalgebra structure. On the other hand we can also see that the map

$$\tau' : \overset{\alpha i_\alpha}{g}_{j_\alpha} \in \mathrm{Pol}(SU_q(2)) \mapsto (-1)^{2\alpha} \overset{\alpha i_\alpha}{g}_{j_\alpha} \in \mathrm{Pol}(SU_q(2)),$$

is also an automorphism. This is a consequence of the identity $\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho} = \oplus_{\gamma=|\alpha-\beta|}^{\alpha+\beta} \overset{\gamma}{\rho}$. Since τ' agrees with τ in $\{a, b, c, d\}$ it follows $\tau' = \tau$. This morphism will be very important for the description of the representations of the quantum Lorentz Group.

7.1.5 Quantum doubles and the Quantum Lorentz Group

All the material presented here is standard. We follow the convention of [BR2] for the Quantum Double. My favourite reference for this subject is [M] chapter 7, where the reader can find the proof of all statements involving quantum Doubles.

Let $\mathcal{A} = (\mathcal{A}, m, 1, \Delta, \varepsilon)$ be a finite dimensional Hopf algebra. Recall the dual vector space $\mathcal{A}$ can also be given a Hopf algebra structure, where all structure maps in $\mathcal{A}^*$ are defined through dualising the ones of $\mathcal{A}$. For example $\Delta : A^* \to A^* \otimes \mathcal{A}^*$ is defined as the composition

$$\mathcal{A}^* \xrightarrow{m^*} (\mathcal{A} \otimes \mathcal{A})^* \xrightarrow{\cong} A^* \otimes A^*,$$

where m^* is the transpose of the multiplicatio $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$. Whereas $(fg)(a) = (f \otimes g)\Delta(a)$, where $f, g \in \mathcal{A}^*$ and $a \in \mathcal{A}$. This product in $\mathcal{A}^*$ is called the convolution product. It makes sense even in $\mathcal{A}$ is infinite dimensional. However the isomorphism $(\mathcal{A} \otimes \mathcal{A})^* \cong \mathcal{A}^* \otimes \mathcal{A}^*$ breaks in the infinite dimensional case, so it is not true that $\mathcal{A}^*$ has a structure of Hopf algebra, for we cannot define a coproduct. However, sometimes the problem can be fixed by restricting to a subalgebra of $\mathcal{A}^*$. This was done in the previous section defining the algebra $\mathrm{Pol}(SU_q(2))$ contained in the dual of $U_q(\mathfrak{su}(2))$.

If $\mathcal{A}$ is an Hopf algebra let A^{cop} be the Hopf algebra with the same multiplication, unit and counit but comultiplication $\Delta^{\mathrm{cop}}(a) = \sum_{(a)} a'' \otimes a'$, thus the antipode of $\mathcal{A}^{\mathrm{cop}}$ is S^{-1}. If $\mathcal{A}$ as a star structure then both A^* and $A^{*\mathrm{cop}}$ have star structures given by $f^*(a) = \overline{f(S^{-1}(a^*))}$ Let $\mathcal{A}$ be a finite dimensional Hopf algebra. The quantum double of $\mathcal{A}$ is denoted by $\mathcal{D}(\mathcal{A}, \mathcal{A}^{*\mathrm{cop}})$. It is $\mathcal{A} \otimes \mathcal{A}^{*\mathrm{cop}}$ as a vector space and coalgebra. However the product in $\mathcal{D}(\mathcal{A}, \mathcal{A}^{*\mathrm{cop}})$ is given by

$$(x \otimes f)(y \otimes g) \sum_{(y),(f)} < f''', S^{-1}(y') >< f', y''' > xy'' \otimes f''g. \tag{137}$$

The antipode of the quantum double is given by $S(a \otimes f) = (1 \otimes S^{-1}(f))(S(a) \otimes 1)$. Here $\Delta(f) = \sum_{(f)} f' \otimes f''$ and $S(f)$ denote the comultiplication and antipode of A^*. Let $\{a_i\}$ be a basis of A and $\{a^i\}$ the dual basis of $\mathcal{A}^*$. Recall the quantum double has an R-matrix given by:

$$R = \sum_i a_i \otimes 1 \otimes 1 \otimes \alpha^i$$

The quantum double also has a star structure given by $(a \otimes f)^* = a^* \otimes f^*$, any time $\mathcal{A}$, thus $\mathcal{A}^*$ has a star structure. This star strucure is always antireal. See [M].

All the identities defining the quantum double as a star Hopf algebra make sense if $\mathcal{A} = U_q(\mathfrak{su}(2))$ and $\mathcal{A}^*$ is substituted by the subalgebra $\mathrm{Pol}(SU_q(2))$ of $U_q(\mathfrak{su}(2))^*$. We can therefore define the quantum Lorentz Group $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ as being $\mathcal{D}(U_q(\mathfrak{su}(2)), \mathrm{Pol}(SU_q(2))^{\mathrm{cop}})$, thus $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ is a star Hopf algebra. For a discussion on why $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ should be considered the quantised enveloping algebra of the Lorentz Group we refer to [BR2] or [PW]. In the last source the dual counterpart of this theory, namely the definition of the algebra of functions in the Quantum Lorentz Group $SL_q(2,\mathbb{C})$ through a double group construction is presented.

7.1.6 Generators and relations for the Quantum Lorentz Group

Recall that the algebra generators of $\mathrm{Pol}(SU_q(2))$ are $\{a, b, c, d\}$ and of $U_q(\mathfrak{su}(2))$ they are $\{q^{\pm J_z}, J_+, J_-\}$. When embedded in the quantum Lorentz Group these generators satisfy see [BR2]:

$$\begin{gathered} q^{J_z}c = qcq^{J_z} \quad q^{J_z}b = q^{-1}bq^{J_z} \quad [q^{J_z}, a] = 0] \quad [q^{J_z}, d] = 0 \quad [J_+, c] = 0 \quad [J_-, b] = 0 \\ [J_+, b] = 0 \quad [J_+, b] = q^{-1}(q^{J_z}a - q^{-J_z}d) \quad [J_-, c] = q(q^{J_z}d - q^{-J_z}a) \\ J_-a = q^{-1}aJ_- + bq^{J_z} \quad aJ_+ = qJ_+a + q^{-J_z}c \quad dJ_- = q^{-1}J_-d + q^{-J_z}b \quad J_+d = qdJ_+. \end{gathered} \tag{138}$$

Therefore the Quantum Lorentz Group is the unital algebra which has generators $\{a, b, c, d, q^{J_z}, q^{J_+}, q^{J_-}, J_+, J_-\}$, the relations defining $U_q(\mathfrak{su}(2))$ and $\mathrm{Pol}(SU_q(2))$, and the mixed relations of the Quantum Double (138). This definition of the quantum Lorentz Group permits us to prove that:

Theorem 96 *There exists an algebra morphism* $\tau : U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}) \to U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ *such that* $\tau(x) = x$ *if* $x \in U_q(\mathfrak{su}(2))$ *and* $\tau(a) = -a, \tau(b) = -b, \tau(c) = -c, \tau(d) = -d$.

This morphism τ had already appeared in 7.1.4. It is of prime importance for the description of the representations of the Quantum Lorentz Group.

Less obvious is the existence of a morphism $\phi : U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}) \to U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$ such that

$$\phi(q^{\pm J_z}) = q^{\pm J_z} \otimes q^{\pm J_z}, \quad \phi(J_\pm) = q^{-J_z} \otimes J_\pm + J_\pm \otimes q^{J_z}, \tag{139}$$

thus ϕ is the comultiplication of $U_q(\mathfrak{su}(2))$; and:

$$\phi(a) = q^{J_z} \otimes q^{-J_z}, \qquad \phi(b) = (1-q^{-2})J_- \otimes q^{-J_z}, \tag{140}$$

$$\phi(c) = (1-q^2)q^{J_z} \otimes J_+, \qquad \phi(d) = q^{-J_z} \otimes q^{J_z} - (q-q^{-1})J_- \otimes J_+. \tag{141}$$

This morphism was defined in [BR2]. One can easily verify that

Theorem 97 *There exists a unique morphism* $\phi : U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}) \to U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$ *which has the form just stated on generators.*

Compare with 2.1. We will give an alternative description for ϕ in 7.2.4, following [BR2]

7.2 An aside on pseudo quasi triangular structures in the algebra $U_q(\mathfrak{su}(2))$

7.2.1 Quasi triangular structure in $U_q(\mathfrak{su}(2))$ and associated knot Invariants

Let $q \in (0,1)$. Recall that $U_q(\mathfrak{su}(2))$ has an R-matrix given by:

$$R = q^{2J_z \otimes J_z} \sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{[n]!}(q-q^{-1})^n (q^{J_z} J_+)^n \otimes (J_- q^{-J_z})^n, \tag{142}$$

with inverse:

$$R^{-1} = q^{-2J_z \otimes J_z} \sum_{n=0}^{\infty} q^{-n} \frac{q^{\frac{n(n+1)}{2}}}{[n]!}(q-q^{-1})^n (q^{2J_z} J_+)^n \otimes (q^{J_z} J_- q^{-J_z})^n. \tag{143}$$

These infinite sums make sense when applied to the tensor product of two finite dimensional representations of $U_q(\mathfrak{su}(2))$, if they are direct sums of representations $\overset{\alpha}{\rho}_w$ where $w=1$. This is because, when considering the action in pair $\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}$ of finite dimensional representations, the infinite sum

$$\sum_{n=0}^{\infty} \frac{q^{\frac{n(n+1)}{2}}}{[n]!}(q-q^{-1})^n (q^{J_z} J_+)^n \otimes (J_- q^{-J_z})^n$$

truncates to a finite sum. Also, by definition $q^{2J_z \otimes J_z}(\overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta}) = q^{2 i_\alpha i_\beta} \overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta}$ if $\overset{\alpha}{v}_{i_\alpha} \in \overset{\alpha}{V}_1 = \overset{\alpha}{V}$ and $\overset{\beta}{v}_{i_\beta} \in \overset{\beta}{V}$. The action of the R-matrix in the other irreducible representations is ambiguous. We say R is an R-matrix of $U_q(\mathfrak{su}(2))$ in the sense that for any pair of representations $\overset{\alpha}{\rho}$ and $\overset{\beta}{\rho}$ of $U_q(\mathfrak{su}(2))$ the map $\overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta} \in \overset{\alpha}{V} \otimes \overset{\alpha}{V} \mapsto R_{21} \overset{\beta}{v}_{i_\beta} \otimes \overset{\alpha}{v}_{i_\alpha} \overset{\beta}{V} \otimes \overset{\alpha}{V}$ is an intertwiner; and that these maps also define a braid group representation. These issues are better described dually, in other words considering the action of the r-form of $\mathrm{Pol}(SU_q(2))$ in the finite dimensional corepresentations of it, as we will see in the next section.

The Hopf algebra $U_q(\mathfrak{su}(2))$ also has a group like element given by $G = q^{2J_z}$. With this formal R-matrix and group like element, the Hopf algebra $U_q(\mathfrak{su}(2))$ is a ribbon Hopf algebra. This means that its category of finite dimensional representations which are direct sums of representations $\overset{\alpha}{\rho}_1 = \overset{\alpha}{\rho}$ is a ribbon category. We thus have a knot invariant $I(\overset{\alpha}{\rho})(q)$ for any $\alpha \in \frac{1}{2}\mathbb{N}_0$ and any $q \in (0,1)$. If K is a framed knot, the assignment $q \in (0,1) \mapsto I(\overset{\alpha}{\rho})(q)(K)$ expresses as a finite sum of square roots of rational functions of q, which are well defined close to 1. Thus they define an analytic function of q in a neighborhood of 1, and in particular a power series in h through the substitution $q = \exp(h)$. Therefore we can interpret these invariants as taking values in $\mathbb{C}[[h]]$. They relate to the coloured Jones Polynomial. In fact, a glance at 1.4.1 tells us immediately that:

Lemma 98 *For any $\alpha \in \frac{1}{2}\mathbb{N}_0$ and $q \in (0,1)$ we have $I(\overset{\alpha}{\rho})(q^{1/2})(K) = J^\alpha(K)(h)$, for any framed knot K. Here $q = \exp(h)$ and $h < 0$*

In other words, the summation of $J^\alpha(K)(h)$ needs to agree with $I(\overset{\alpha}{\rho})(q^{1/2})(K)$ for $q \in (0,1)$ and close enough to 1. Recall $J^\alpha(K)(h)$ is a power series with an infinite radius of convergence.

7.2.2 Corepresentations of $\mathrm{Pol}(SU_q(2))$ and r-form

A good reference now is [M]. As we have mentioned in the previous section, the problem arising from the fact the R-matrix of $U_q(\mathfrak{su}(2))$ not precisely an R-matrix can be eliminated by considering the dual picture. Let $\mathcal{A}$ be a Hopf algebra. In particular we can consider the convolution product in $\mathcal{A}^*$ such that $(fg)(a) = (f \otimes g)\Delta(a)$. Its identity is $\varepsilon : \mathcal{A} \to \mathbb{C}$. An element $r \in (\mathcal{A} \otimes A)^*$ is an r-form if it has a convolution inverse $\bar{r}$ and:

1.
$$\sum_{(a)(b)} r(a' \otimes b')a''b'' = \sum_{(a)(b)} r(a'' \otimes b'')b'a', \forall a, b \in \mathcal{A},$$

2.
$$r(ab, c) = \sum_{(c)} r(a, c')r(b, c''), \forall a, b, c \in \mathcal{A},$$

3.
$$r(a, bc) = \sum_{(a)} r(a', c)r(a'', b), \forall a, b, c \in \mathcal{A}.$$

Consider the bilinear form r in $\mathrm{Pol}(SU_q(2))$ given by

$$r(\overset{\alpha i_\alpha}{g}_{j_\alpha}, \overset{\beta i_\beta}{g}_{j_\beta}) = \left\langle \overset{\alpha i_\alpha}{v} \otimes \overset{\beta i_\beta}{v} \left| (\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho})(R) \right| \overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta} \right\rangle, \tag{144}$$

then r is an r-form of $\mathrm{Pol}(SU_q(2))$. See [KS], 8.3.3 and 10.1. Its convolution inverse is:

$$r^{-1}(\overset{\alpha i_\alpha}{g}_{j_\alpha}, \overset{\beta i_\beta}{g}_{j_\beta}) = \left\langle \overset{\alpha i_\alpha}{v} \otimes \overset{\beta i_\beta}{v} \left| (\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho})(R^{-1}) \right| \overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta} \right\rangle. \tag{145}$$

In particular given two corepresentations ϕ and ψ of $\mathrm{Pol}(SU_q(2))$ in V and W there exists an intertwiner $V \otimes W \to W \otimes V$. It has the form

$$V \otimes W \xrightarrow{x \otimes y \mapsto y \otimes x} W \otimes V \xrightarrow{\psi \otimes \phi} W \otimes \mathrm{Pol}(SU_q(2)) \otimes V \otimes \mathrm{Pol}(SU_q(2)) \xrightarrow{r_{24}} W \otimes V. \tag{146}$$

Consider the corepresentations $\psi_a, \alpha \in \frac{1}{2}\mathbb{N}_0$ of $\mathrm{Pol}(SU_q(2))$ in $\overset{\alpha}{V}$ given by:

$$\overset{\alpha}{v}_{i_\alpha} \mapsto \overset{\alpha}{v}_{j_\alpha} \otimes \overset{\alpha j_\alpha}{g}_{i_\alpha},$$

then obviously the intertwiner in (146) is exactly $v \otimes w \mapsto R_{21} w \otimes v$. This is probably the clearest way to understand the R-matrix in $U_q(\mathfrak{su}(2))$ in a precise level when considering representations $\overset{\alpha}{\rho}_1$ (though not the clearest in the intuitive level). Analogously, the ribbon category structure in the category of representations of

$U_q(\mathfrak{su}(2))$ which are direct sums of representations $\overset{\alpha}{\rho}$ can be understood by considering the coribbon structure in $\mathrm{Pol}(SU_q(2))$, dual to the heuristic ribbon structure of $U_q(\mathfrak{su}(2))$ described for example in [BR2].

It is well know that the corepresentations ψ_α exhaust all the irreducible corepresentation of $\mathrm{Pol}(SU_q(2))$ and that any corepresentation of it expresses as a direct sum of these. Therefore, the representations $\overset{\alpha}{\rho}_w$ of $U_q(\mathfrak{su}(2))$ with $w \neq 1$ are not dual to any correpresentation of $\mathrm{Pol}(SU_q(2))$. This is the reason why the triangular and ribbon structures in the representations $\overset{\alpha}{\rho}_w$ with $w = 1$ do not extend to all finite dimensional representations of $U_q(\mathfrak{su}(2))$.

The algebra $U_q(\mathfrak{su}(2))$ is a subalgebra of the dual $\mathrm{Pol}(SU_q(2))^*$ of $\mathrm{Pol}(SU_q(2))$, the last given the convolution product. It is not difficult to verify that:

Proposition 99 *For any $\alpha \in \frac{1}{2}\mathbb{N}_0$ and any $-\alpha \leq i_\alpha, i_\beta \leq \alpha$ the map*

$$f \in \mathrm{Pol}(SU_q(2)) \mapsto r(\overset{\alpha i_\alpha}{g}_{j_\alpha}, f) \in \mathbb{C}$$

is in $U_q(\mathfrak{su}(2))$, considering the embedding $U_q(\mathfrak{su}(2)) \to \mathrm{Pol}(SU_q(2))^*$. Therefore it defines a map $\mathrm{Pol}(SU_q(2)) \to U_q(\mathfrak{su}(2))$ which we denote by: $g \mapsto (g \otimes \mathrm{id})(R)$.

We thus have $< \overset{\beta i_\beta}{v} | (\overset{\alpha i_\alpha}{g}_{j_\alpha} \otimes \mathrm{id})(R) \overset{\beta}{v}_{i_\beta} > = < (\overset{\alpha i_\alpha}{g}_{j_\alpha} \otimes \mathrm{id})(R), \overset{\beta i_\beta}{g}_{j_\beta} > = r(\overset{\alpha i_\alpha}{g}_{j_\alpha}, \overset{\beta i_\beta}{g}_{j_\beta})$, by definition. Actually, this map is a morphism. This is because for any $f \in \mathrm{Pol}(SU_q(2))$ we have:

$$\begin{aligned}
< (gh \otimes \mathrm{id})(R), f > &= r(gh, f) \\
&= \sum_{(f)} r(g, f')r(h, f'') \\
&=< (g \otimes \mathrm{id})(R), f' >< (h \otimes \mathrm{id})(R), f'' > \\
&=< (g \otimes \mathrm{id})(R)(h \otimes \mathrm{id})(R), f > .
\end{aligned}$$

We can derive some other identities, which will be useful in 7.3.8. Let $f, g, h \in \mathrm{Pol}(SU_q(2))$. We thus have

$$\begin{aligned}\sum_{(f)} < (f' \otimes \mathrm{id})(R), g >< (f'' \otimes \mathrm{id})(R), h > &= \sum_{(f)} r(f', g) r(f'', h) \\ &= r(f, hg) \\ &=< (f \otimes \mathrm{id})(R), hg > .\end{aligned}$$

Let $R^{(-)} = R_{21}^{-1}$. If $f \in \mathrm{Pol}(SU_q(2))$, we can define analogously $(f \otimes \mathrm{id})(R^{(-)}) \in U_q(\mathfrak{su}(2))$. In particular $< (f \otimes \mathrm{id})(R^{(-)}, g >= r_{21}^{-1}(f, g) = \hat{r}(f, g)$. Let $f, g, h \in \mathrm{Pol}(SU_q(2))$. We have

$$\begin{aligned}\sum_{(f)} < (f' \otimes \mathrm{id})(R), g >< (f'' \otimes \mathrm{id})(R^{(-)}), h > &= \sum_{(f)} r(f', g) \hat{r}(f'', h) \\ &= r_{12} \hat{r}_{13}(f, g, h)\end{aligned}$$

.

The last equality follows by definition of the convolution product in $\mathrm{Pol}(SU_q(2))^*$.

7.2.3 Quantum co-double of quasitriangular Hopf algebras and knot invariants

The best reference now for general quantum group theory is still [M], chapter 7. Let $\mathcal{A}$ be a Hopf algebra with an R-matrix R. The quantum co-double $\mathcal{A} \hat{\otimes} \mathcal{A}$ of $\mathcal{A}$ is defined as the Hopf algebra which has $\mathcal{A} \otimes \mathcal{A}$ has underlying algebra and with counit $\varepsilon \otimes \varepsilon$. The modified coproduct is:

$$\Delta(a \otimes b) = R_{23}^{-1} a' \otimes b' \otimes a'' \otimes b'' R_{23},$$

whereas the antipode is defined as:

$$S(a \otimes b) = R_{21} S(a) \otimes S(b) R_{21}^{-1}.$$

If $\mathcal{A}$ is a star Hopf algebra, this is also a star Hopf algebra with the following definition of star: $(a \otimes b)^* = R_{21} b^* \otimes a^* R_{21}^{-1}$. See [KS] 10.2.4 or [M] . Notice that we can take any element F of $\mathcal{A} \otimes \mathcal{A}$ with $(\Delta \otimes \mathrm{id})(F) = F_{13}F_{23}$ and $(\mathrm{id} \otimes \Delta)(F) = F_{13}F_{12}$, defining always a Hopf algebra. The advantage of picking R is that the Hopf algebra $\mathcal{A} \hat{\otimes} \mathcal{A}$ automatically has an R-matrix[11] given by

$$\hat{R} = R_{14}^{(-)} R_{24}^{(-)} R_{13}^{(+)} R_{23}^{(+)},$$

where $R^{(+)} = R$ and $R^{(-)} = R_{21}^{-1}$. Let $u = \sum_i S(t_i) s_i$ and $\hat{u} = \sum_i S(\hat{t}_i) \hat{s}_i$ for $R = \sum_i s_i \otimes t_i$ and $\hat{R} = \sum_i \hat{s}_i \otimes \hat{t}_i$. It is easy to prove that $\hat{u} = u \otimes S(u^{-1})$. We always have $S^2(u) = u$, thus if $\mathcal{A}$ is a ribbon Hopf algebra with ribbon element θ then the quantum co-double of $\mathcal{A}$ will also be a ribbon Hopf algebra with ribbon element $\hat{\theta} = \theta \otimes \theta^{-1}$. The group like element $\hat{u}\hat{\theta}$ of $\mathcal{A} \hat{\otimes} \mathcal{A}$ will thus be $\hat{G} = \hat{u}\hat{\theta}^{-1} = G \otimes G$. Notice $\theta^2 = uS(u)$ and $S(\theta) = \theta$ thus $S(u^{-1})\theta = u\theta^{-1} = G$, since θ is central. All this is referred to in [BNR].

Let $\mathcal{A}$ be a ribbon Hopf algebra. Let $\overset{\alpha}{\rho}$ and $\overset{\beta}{\rho}$ be two finite dimensional representations of $\mathcal{A}$, which will then generate a representation $\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}$ of $\mathcal{A} \hat{\otimes} \mathcal{A}$. Unpacking the expression of the $\mathcal{A} \hat{\otimes} \mathcal{A}$ framed knot invariants coming out of the representation $\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}$ yields immediately

Lemma 100 *For any framed knot K we have*

$$I(\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho})(K) = I(\overset{\alpha}{\rho})(K) I(\overset{\beta}{\rho})(K^*),$$

[11] There are two obvious possible choices of an R-matrix in the quantum codouble. If R is real then one of them is real and the other one anti real. See [M]. We have picked the antireal one. Since the R-matrix of $U_q(\mathfrak{su}(2))$ is real, this is in acordance with 7.1.1

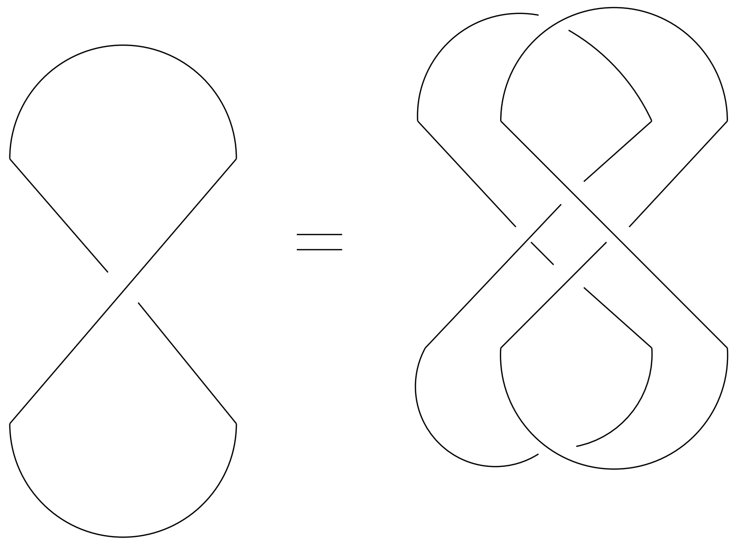

Figure 30: A knot K and $K^* \cup K$

where K^* is the mirror image of K.

This is because $I(\overset{\alpha}{\rho})(K)I(\overset{\beta}{\rho})(K^{)})$ equals the $\mathcal{A}$-knot invariant evaluated in $K \cup K^*$ for K coloured with $\overset{\alpha}{\rho}$ and K^* coloured with $\overset{\beta}{\rho}$. We suppose that K^* is placed away from K as in figure 30. It is also important to remember that $S(a \otimes b) = R_{21} S(a) \otimes S(b) R_{21}^{-1}$ in the quantum co-double. This lemma is used implicitly in [BR2] and [BNR] for several times. For the ones who want more details: A good way to prove this lemma precisely is to use the fact that if a knot is the closure of a braid b with n strands and ρ is a representation in V, then $I(\rho)(K) = \mathrm{tr}\{v \in V^{\otimes n} \mapsto G^{\otimes n} R(b)(v)\}$, where $R(b)$ is the braiding operator $V^{\otimes n} \to V^{\otimes n}$. This expression works for any ribbon Hopf algebra. Therefore the lemma is obvious from the form of the R-matrix of $\mathcal{A} \hat{\otimes} \mathcal{A}$ is $\hat{R} = R_{14}^{(-)} R_{24}^{(-)} R_{13}^{(+)} R_{23}^{(+)}$ and the fact $\hat{G} = G \otimes G$.

In the $U_q(\mathfrak{su}(2))$ case things are slightly more complicated. We can define the coevaluation map by considering the action of $U_q(\mathfrak{su}(2))$ in its finite dimensional representations. We can do the same for the antipodal map. The point is that tensor products and duals of representations of $U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$ are perfectly well defined. Thus formally $U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$ is still a ribbon Hopf algebra, in the sense that the representations of it which are direct sums of representations $\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}$ is a ribbon category. This follows from the fact $U_q(\mathfrak{su}(2))$ with its formal R-matrix and group like element G is a ribbon category. The group like element of $U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$ is $G \otimes G$, where $G = q^{2J_z}$ is the group like element of $U_q(\mathfrak{su}(2))$. A good way to present this arguments in a cleaner way is again to consider the dual picture. Namely the dual construction with the r-form of $\mathrm{Pol}(SU_q(2))$ yields an Hopf algebra isomorphic with $\mathrm{Pol}(SU_q(2)) \otimes \mathrm{Pol}(SU_q(2))$ as a co-algebra, whereas the product is the generalised quantum double [12] relative to the pairing $r : \mathrm{Pol}(SU_q(2)) \otimes \mathrm{Pol}(SU_q(2)) \to \mathbb{C}$. This Hopf algebra has an r-form given by $r_{41}^{-1}r_{24}^{-1}r_{13}r_{23}$. See [KS] 10.2.3 [13].

We can calculate knot invariants from $U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$, and any representations $\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}$ of it. Denote this invariant by $I(\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho})$. By lemmas 98 and 100 it follows that:

Lemma 101 *For any $\alpha, \beta \in \frac{1}{2}\mathbb{N}_0$ we have:*

$$I(\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho})(q^{1/2}) = L^{\alpha,\beta}(K)(-h),$$

where $q = (0,1)$ and $\exp(h) = q$ for $h < 0$.

Recall 3.3.3 and that $L^{\alpha,\beta}(K)(q)$ is a convergent power series if $\alpha, \beta \in \frac{1}{2}\mathbb{N}_0$. Notice

[12] The simpler case considered before was made out of the canonical dual pairing between an algebra and its dual

[13] Notice similarly that this generalised quantum double has two natural r-forms

that in the notation of chapter 2 we have:

$$L^{\alpha,\beta}(K)(h) = X(\alpha-\beta, \alpha+\beta+1)(2\alpha+1)(2\beta+1)$$

from Theorem 16.

7.2.4 The quantum co-double of $U_q(\mathfrak{su}(2))$ and the Quantum Lorentz Group

Suppose $\mathcal{A}$ is a finite dimensional Hopf algebra with an R-matrix R. There exists a natural vector space map $\phi : \mathcal{D}(A, A^{*\mathrm{cop}}) \mapsto \mathcal{A}\hat{\otimes}\mathcal{A}$. It has the form:

$$\phi : (x \otimes f) \mapsto \sum_{(x)(f)} x'(f'' \otimes \mathrm{id})(R^{(+)}) \otimes x''(f' \otimes \mathrm{id})(R^{(-)}) = \phi(x)\phi(y). \tag{147}$$

The coevaluations are taken in $\mathcal{A}$ and $\mathcal{A}^*$ (not in $\mathcal{A}^{*\mathrm{cop}}$). The map ϕ transforms the R-matrix of the quantum double in the R-matrix of $\mathcal{A}\hat{\otimes}\mathcal{A}$. This is an easy consequence of the fact $(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23} = (\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}$ for any R-matrix R, from which follows $(\Delta \otimes \mathrm{id})(R^{(-)}) = R^{(-)}_{13}R^{(-)}_{23}$. Indeed let $R = \sum_i s_i \otimes t_i$ and $R^{-1} = \sum_i \bar{s}_i \otimes \bar{t}_i$. The R-matrix of the quantum double is $\sum_k X_k \otimes 1 \otimes 1 \otimes X^k$ where $\{X_k\}$ is a basis of $\mathcal{A}$ and $\{X^k\}$ its dual basis of $\mathcal{A}^*$. We have

$$\begin{aligned}
(\phi\otimes\phi)(R) &= (\phi\otimes\phi)(\sum_k X_k \otimes 1 \otimes 1 \otimes X^k) \\
&= \sum_{k,(X^k),i,j} \Delta(X_k) \otimes {X^k}''(s_i)t_i \otimes {X^k}'(\bar{t_j})\bar{s_j} \\
&= \sum_{i,j,k} \Delta(X_k) \otimes X^k(\bar{t_j}s_i)t_j \otimes \bar{s_j} \\
&= (\Delta \otimes \mathrm{id})(R^{(-)}_{13}R^{(+)}_{12}) \\
&= R^{(-)}_{14}R^{(-)}_{24}R^{(+)}_{13}R^{(+)}_{23}.
\end{aligned}$$

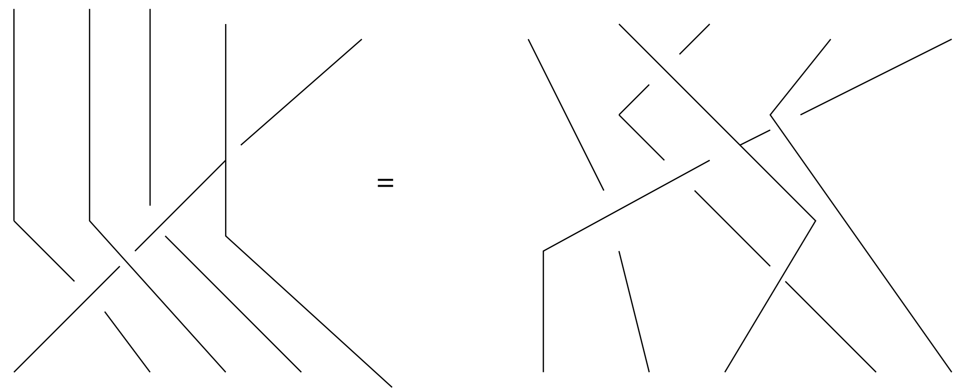

Figure 31: An equality between two braids

For the third equality, notice that $\mathcal{A}$ is isomorphic with $\mathcal{A}^{**}$, the last given the convolution product. In particular $< a \otimes b, \Delta(f) >=< ab, f >, \forall a, b \in \mathcal{A}, \forall f \in \mathcal{A}^*$. The map ϕ is always a morphism of Hopf algebras algebras. This is shown in [M] (Theorem 7.3.5), for instance. It is not totally trivial. For the comultiplication for example, the difficult part is to prove that $\Delta(\phi(g)) = (\phi \otimes \phi)(\Delta(g))$, if $g \in \mathcal{A}^*$. Let $R = \sum_i s_i \otimes t_i$ and $R^{-1} = \sum_i \hat{s_i} \otimes \hat{t_i}$. Notice $\phi(g) = \sum_{i,j} g(\hat{t_j} s_i) t_i \otimes \hat{s_j}$. We have (notice it is the co-algebra $\mathcal{A}^{*\mathrm{cop}}$ which is embedded in $\mathcal{D}$, not $\mathcal{A}$, thus $\Delta_{\mathcal{D}}(g) = \sum_{(g)} g'' \otimes g'$):

$$\Delta(\phi(g)) = \sum_{i,j,k,l,(t_i),(\hat{s_j})} g(\hat{t_j} s_i) t_i' \otimes \hat{s_k} \hat{s_j}' s_l \otimes \hat{t_k} t_i'' t_l \otimes s_j'',$$

whereas

$$(\phi \otimes \phi)\Delta(g) = \sum_{i,j,k,l} g(\hat{t_k} s_l \hat{t_j} \hat{s_i}) t_i \otimes \hat{s_j} \otimes \hat{t_l} \otimes \hat{s_k}$$

The fact the two expressions are equal is a consequence of Mac Lane's coherence theorem for braided tensor categories. See figure 31.

The identities $(S \otimes S)(R) = R$ and $(S \otimes S) \circ \Delta = \Delta^{\mathrm{cop}} \circ S$, and a similar argument proves ϕ intertwines S as well. Notice $S_{\mathcal{D}} = S^{-1}$ when restricted to $\mathcal{A}^*$.

In the $U_q(\mathfrak{su}(2))$ case, since it is not a braided Hopf algebra in the strict sense, things are slightly trickier. However, the map ϕ still makes sense, from proposition 99 and comments after[14]. Moreover, if restricted to $U_q(\mathfrak{su}(2))$ or to $\mathrm{Pol}(SU_q(2))$ it is an algebra morphism. Indeed, in the $U_q(\mathfrak{su}(2))$ case this is because ϕ is simply the comultiplication, whereas in the $\mathrm{Pol}(SU_q(2))$-case, this was proved after Proposition 99. Let us see it is a morphism of algebras. We only need to show that ϕ satisfies the mixed relations of the Quantum Double in 7.1.6. This since ϕ when restriced to $\mathrm{Pol}(SU_q(2))$ and $U_q(\mathfrak{su}(2))$ is a morphism follows if relations (138) are satisfied. We can explicitely calculate ϕ in the generators of the Quantum Lorentz Group. In fact we trivially have:

$$\phi(X) = \Delta(X), X \in U_q(\mathfrak{su}(2)), \tag{148}$$

and, recall (136):

$$\begin{gathered} a \mapsto q^{J_z} \otimes q^{-J_z}, b \mapsto (1-q^{-2})J_- \otimes q^{-J_z} \\ c \mapsto (1-q^2)q^{J_z} \otimes J_+, d \mapsto q^{-J_z} \otimes q^{J_z} - (q-q^{-1})J_- \otimes J_+. \end{gathered} \tag{149}$$

These are exactly the values of the morphism ϕ defined 7.1.6 in the generators of the Quantum Lorentz Group. In particular ϕ as defined by (147) does define an algebra morphism equivalent to it. Indeed, let ϕ' be the morphism defined in 7.1.6. We have

$$\phi(x \otimes f) = \phi(x)\phi(f) = \phi'(x)\phi'(f) = \phi'(xf) = \phi'(x \otimes f).$$

The first equalily by definition of ϕ, the second because ϕ and ϕ' are algebra morphisms when restricted to $\mathrm{Pol}(SU_q(2))$ and $U_q(\mathfrak{su}(2))$, which are equal in generating sets of $\mathrm{Pol}(SU_q(2))$ and $U_q(\mathfrak{su}(2))$, the third because ϕ' is a morphism the forth by

[14]Majid proves this is full generality. See [M], example 7.3.6.

the definition of multiplication in the Quantum Double. Thus it follows $\phi = \phi'$ and since ϕ' is a morphism ϕ is as well.

The existence of this Hopf algebra morphism ϕ was stated in [BR2], however in the way it is stated it is not clear that everything works for we only have an heuristic R-matrix in $U_q(\mathfrak{su}(2))$. This construction is one of the major tools for proving the main theorems of this chapter, the reason why I presented it here with all this detail. This is also proved in [M], as we have referred to before.

The morphism $\phi : U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}) \to U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$ gives us a representation of the Quantum Lorentz Group for any pair of representation of $U_q(\mathfrak{su}(2))$. We shall go back to this in 7.3.1.

7.3 Representations of the Quantum Lorentz Group

We now look at the representation theory of the quantum Lorentz group. We base our approach on examples, with the final aim of describing the knot theory coming out of them. The main reference is [BR2].

7.3.1 Crossed $\mathrm{Pol}(SU_q(2))$-bimodules and finite dimensional representations of the Quantum Lorentz Group

A class of finite dimensional representations of the quantum Lorentz Group arises if we consider $\mathrm{Pol}(SU_q(2))$-crossed bimodules. A crossed bimodule is a finite dimensional complex vector space V provided with a left representation of $\mathrm{Pol}(SU_q(2))$ and a right corepresentation of $\mathrm{Pol}(SU_q(2))$. They need to be compactible in the sense:

$$\sum_{(a)(v)} a' v_V \otimes a''(v_{\mathrm{Pol}(SU_q(2))}) = \sum_{(a)(a''v)} (a''v)_V \otimes (a''v)_{\mathrm{Pol}(SU_q(2))} a'. \tag{150}$$

The notation $v \in V \mapsto \sum_{(v)} v_V \otimes v_{\mathcal{A}} \in V \otimes \mathcal{A}$ stands for a corepresentation of the bialgebra $\mathcal{A}$ in the vector space V. The following is a result of [T], proposition 5.1[15]:

Proposition 102 *Let V be a finite dimensional vector space equipped with a left representation ρ_1 and a right representation ρ_2' of $\mathrm{Pol}(SU_q(2))$. The right corepresentation ρ_2' of $\mathrm{Pol}(SU_q(2))$ yields a left representation ρ_2 of $U_q(\mathfrak{su}(2))$ in V in the usual fashion: $av = \sum_{(v)} v_V \left\langle a, v_{\mathrm{Pol}(SU_q(2))}\right\rangle$ where $a \in U_q(\mathfrak{su}(2)), v \in V$. The pair (ρ_1, ρ_2') is a crossed bimodule if and only if $\rho_2 \otimes \rho_1$ is a representation of the Quantum Lorentz Group $\mathcal{D}(U_q(\mathfrak{su}(2)), \mathrm{Pol}(SU_q(2)) = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$.*

A crossed bimodule is called simple if it yields an irreducible representation of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$.

Recall the morphism $\tau : \mathrm{Pol}(SU_q(2)) \to \mathrm{Pol}(SU_q(2))$ defined in the end of 7.1.4. As proved in 7.1.6, it extends to an algebra morphism $\tau : U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}) \to U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$, in the fashion $X \mapsto X$ for all $X \in U_q(\mathfrak{su}(2))$. See also [BR2] page 507 and [T] page 571. Recall also the algebra morphism $\phi : U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}) \to U_q(\mathfrak{su}(2)) \otimes U_q(\mathfrak{su}(2))$, defined in 7.1.6 and 7.2.4. A main result of [T] is:

Theorem 103 *Let V be a simple finite dimensional crossed module of $\mathrm{Pol}(SU_q(2))$. There exists $\alpha, \beta \in \frac{1}{2}\mathbb{N}_0$ such that either ρ of the form $(\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}) \circ \phi$ or ρ is of the form $(\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}) \circ \phi \circ \tau$.*

Notice that (148) and (149) together with the explicit form of the representation $\overset{\alpha}{\rho}$

[15]In Takeuchi's convention the product in $\mathrm{Pol}(SU_q(2))$ is oposite of the one we consider

permits us to describe explicitely $(\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}) \circ \phi$ and $(\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}) \circ \phi \circ \tau$. Compare with [T], equation (6.4). See also [BR2] page 507.

Notice that not all representations of $U_q(\mathfrak{su}(2))$ are dual to corepresentations of $\mathrm{Pol}(SU_q(2))$. This happens if, and only if, they are direct sums of representations $\overset{\alpha}{\rho}$ with $a \in \frac{1}{2}\mathbb{N}_0$. Therefore Theorem 103 classifies all finite dimensional irreducible representations of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ which, when restricted to $U_q(\mathfrak{su}(2))$, are a direct sum of representations $\overset{\alpha}{\rho}, \alpha \in \frac{1}{2}\mathbb{N}_0$. These representations are the ones which are dual to corepresentation of the algebra of function in the Quantum Lorentz Group, in other words the quantum group $SL_q(2,\mathbb{C})$ of [PW].

Definition 104 *These kind of representations of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ are called physical*[16] *representations of $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$.*

Definition 105 *Any finite dimensional representation of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ generates a finite dimensional representation of $U_q(\mathfrak{su}(2))$ through the embedding $U_q(\mathfrak{su}(2)) \subset U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$, therefore any finite dimensional representation ρ of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ decomposes as a direct sum of $\overset{\alpha}{\rho}_{w_\alpha}$'s with each a in $\frac{1}{2}\mathbb{N}_0$ and each $w_\alpha \in \{1,-1,i,-i\}$, uniquely. In addition, if the representation is physical then we always have $w_\alpha = 1$ for any α. The smallest of these α's is by definition the minimal spin of ρ. The maximal spin of ρ is the maximum of those.*

From the fact $\phi(X) = \phi(X) \circ \tau = \Delta(X)$ for any $X \in U_q(\mathfrak{su}(2))$ and Theorem 103 it is immediate that

Theorem 106 *For any α in $\frac{1}{2}\mathbb{N}_0$ and the minimal spin of $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi$ and $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau$,*

[16] This is my nomenclature

is zero. Moreover, any irreducible physical representation ρ of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ with minimal spin zero is of this form, in fact α is half the maximal spin of ρ.

7.3.2 An equation producing representations

We now give a general form of representations of the Quantum Lorentz Group. Let $\mathcal{D} = \mathcal{D}(U_q(\mathfrak{su}(2)), \mathrm{Pol}(SU_q(2))^{\mathrm{cop}}) = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ be the Quantum Lorentz Group at $q \in (0,1)$. Recall equation (137) expressing the product law in the quantum double. Let $a = a \otimes 1 \in \mathcal{D}$ and $f = 1 \otimes f \in \mathcal{D}$, where $a \in U_q(\mathfrak{su}(2))$ and $f \in \mathrm{Pol}(SU_q(2))$. We thus have $af = a \otimes f, a \in U_q(\mathfrak{su}(2)), f \in \mathrm{Pol}(SU_q(2))$. Let us consider representations ρ of $\mathcal{D}$ of the vector space

$$V = \bigoplus_{\alpha \in \mathbb{N}_0} \overset{\alpha}{V},$$

where

$$\overset{\alpha}{\rho} : U_q(\mathfrak{su}(2)) \to L(\overset{\alpha}{V})$$

is the irreducible representation of $U_q(\mathfrak{su}(2))$ with spin α. Compare with 2.1.1. Notice we are only considering integer spins. The case in which we admit any spins is similar. It is natural to consider representations in V since through the embedding $U_q(\mathfrak{su}(2)) \to U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ any finite dimensional representation of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ decomposes as direct sums of representations $\overset{\alpha}{\rho}$ when restricted to $U_q(\mathfrak{su}(2))$. We obvioulsy suppose that any element x of $U_q(\mathfrak{su}(2))$ acts in V in the fashion:

$$\rho(x) = \prod_{\alpha \in \mathbb{N}_0} \overset{\alpha}{\rho}(x). \tag{151}$$

A basis of $\overset{\alpha}{V}$ is given by the vectors $\{\overset{\alpha}{u}_i, i = -\alpha, -\alpha+1, ..., \alpha\}$. Consider an action of $\mathcal{D}$ in V such that $\mathrm{Pol}(SU_q(2))$ acts as

$$\overset{\alpha i_\alpha \beta}{g_{j_\alpha}} \overset{\beta}{v}_{i_\beta} = \frac{\mathcal{F}_\beta}{\mathcal{F}_\gamma} \sum_{D,\gamma\ \in\mathbb{N}_0} \sum_{\substack{-\gamma\le i_\gamma\le\gamma \\ -D\le x\le D}} \overset{\gamma}{v}_{i_\gamma} \begin{pmatrix} i_\gamma & i_\alpha & \Big| & D \\ \gamma & \alpha & \Big| & x \end{pmatrix} \begin{pmatrix} x & \Big| & \alpha & \beta \\ D & \Big| & j_\alpha & i_\beta \end{pmatrix} \Lambda^{\alpha D}_{\gamma\beta}. \qquad (152)$$

The compatibility rules for the spins ensure it is a finite sum and that $\phi(f)V \subset V$ if $f \in \mathrm{Pol}(SU_q(2))$. Extend it to all $\mathcal{D}$ as $\rho(a\otimes f) = \rho(a)\rho(f)$. Let us describe which constraints the constants $\Lambda^{\beta\gamma}_{\alpha\delta}$ need to satisfy. It is obvious that $\rho(a)\rho(b) = \rho(ab)$ if $a \in U_q(\mathfrak{su}(2))$ and $b \in \mathcal{D}$. One can also show that automatically we have $\rho(fa) = \rho(f)\rho(a)$ if $f \in \mathrm{Pol}(SU_q(2))$ and $a \in U_q(\mathfrak{su}(2))$. This calculation appears in [BR2], inside the proof of theorem 3. Therefore the only restriction put in the constants Λ^{BC}_{AD} is that (152) define a representation of $\mathrm{Pol}(SU_q(2))$.

The following is a result of [BR2]:

Theorem 107 *If*

$$\sum_E \begin{Bmatrix} A & E & \Big| & C \\ B & U & \Big| & D \end{Bmatrix} \Lambda^{AC}_{FE}\Lambda^{BD}_{EP} = \sum_K \begin{Bmatrix} F & A & \Big| & C \\ B & U & \Big| & K \end{Bmatrix} \begin{Bmatrix} A & B & \Big| & K \\ P & U & \Big| & D \end{Bmatrix} \Lambda^{KU}_{FP}, \quad (153)$$

then (151) and (152) define a representation of the Quantum Lorentz Group $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$.

A proof of this is not too difficult, but requires some additional notation and calculations, so will be omitted. The reader can find it in [BR2] inside the proof of theorem 3. All this is to be compared with 2.1.1 and especially (7).

One observation in need concerns the constants $\mathcal{F}_\alpha, a \in \mathbb{N}$. Their values are (almost) arbitrary and they only appear to make some of the representations unitary. In fact their appearance does not change the representation apart from isomorphism therefore not affecting the future calculation of knot invariants.

7.3.3 A solution of (153) and balanced representations of the Quantum Lorentz Group

In [BR2], Buffenoir and Roche describe a solution of (153) for any $p \in \mathbb{C}$. This Λ_{AD}^{BC}-coefficients are defined out of a complicated complex continuation of $6j$-symbols, but in a way we can easily prove that the constraints (153) are satisfied. At the end they are proved to be equal to:

$$\Lambda_{AD}^{BC}(p) = \sum_{\sigma} \begin{pmatrix} 0 & \Big| & C & B \\ A & \Big| & \sigma & -\sigma \end{pmatrix} q^{2\sigma p} \begin{pmatrix} -\sigma & \sigma & \Big| & D \\ B & C & \Big| & 0 \end{pmatrix}. \tag{154}$$

We need to choose a logarithm of q, so the best notation would be to consider $q = \exp(h)$. However for $q \in (0,1)$ we have no big ambiguity. We always pick $h < 0$, in accordance with before. For completeness, and because this is not show directly in [BR2] we prove that the coefficients Λ_{AD}^{BC} verify (153), for any $p \in \mathbb{C}$. Our main results depend on this fact, ultimately. The proof goes as:

$$\sum_{E} \begin{Bmatrix} A & E & \Big| & C \\ B & U & \Big| & D \end{Bmatrix} \Lambda_{FE}^{AC} \Lambda_{EP}^{BD}$$

$$= \sum_{E} \begin{Bmatrix} A & E & \Big| & C \\ B & U & \Big| & D \end{Bmatrix} \sum_{a,b} q^{2(a+b)p}$$

$$\begin{pmatrix} 0 & \Big| & C & A \\ F & \Big| & a & -a \end{pmatrix} \begin{pmatrix} -a & a & \Big| & E \\ A & C & \Big| & 0 \end{pmatrix} \begin{pmatrix} 0 & \Big| & D & B \\ E & \Big| & b & -b \end{pmatrix} \begin{pmatrix} -b & b & \Big| & P \\ B & D & \Big| & 0 \end{pmatrix}$$

$$= \sum_{E} \begin{Bmatrix} A & E & \Big| & C \\ B & U & \Big| & D \end{Bmatrix} \sum_{a,b} q^{2(a+b)p}$$

$$\begin{pmatrix} 0 & \Big| & C & A \\ F & \Big| & a & -a \end{pmatrix} \begin{pmatrix} b & -b & \Big| & E \\ D & B & \Big| & 0 \end{pmatrix} \begin{pmatrix} 0 & \Big| & A & C \\ E & \Big| & -a & a \end{pmatrix} \begin{pmatrix} -b & b & \Big| & P \\ B & D & \Big| & 0 \end{pmatrix}.$$

The last equation from (112). Since, from the last bit of(124), we have

$$\left\{\begin{matrix} A & E \\ B & U \end{matrix} \middle| \begin{matrix} C \\ D \end{matrix}\right\} = \left\{\begin{matrix} A & U \\ B & E \end{matrix} \middle| \begin{matrix} D \\ C \end{matrix}\right\},$$

by (122), (110) and (111) we can write this as:

$$= \sum_{a,b} q^{2(a+b)p}$$
$$\begin{pmatrix} 0 & \Big| & C & A \\ F & \Big| & a & -a \end{pmatrix} \begin{pmatrix} b & \Big| & A & U \\ D & \Big| & -a & a+b \end{pmatrix} \begin{pmatrix} a+b & -b & \Big| & C \\ U & B & \Big| & a \end{pmatrix} \begin{pmatrix} -b & b & \Big| & P \\ B & D & \Big| & 0 \end{pmatrix},$$

which using equation (112) equals:

$$\sum_{a,b} q^{2(a+b)p}$$
$$\begin{pmatrix} 0 & \Big| & C & A \\ F & \Big| & a & -a \end{pmatrix} \begin{pmatrix} -a & a+b & \Big| & D \\ A & U & \Big| & b \end{pmatrix} \begin{pmatrix} a & \Big| & U & B \\ C & \Big| & a+b & -b \end{pmatrix} \begin{pmatrix} -b & b & \Big| & P \\ B & D & \Big| & 0 \end{pmatrix}.$$

This term can be written as, again by (122), (110) and (111) :

$$\sum_{a,b} q^{2(a+b)p} \sum_{M,N} \begin{Bmatrix} C & A & \big| & F \\ U & M & \big| & D \end{Bmatrix} \begin{Bmatrix} U & B & \big| & C \\ D & N & \big| & P \end{Bmatrix}$$

$$\begin{pmatrix} 0 & a+b & \big| & M \\ F & U & \big| & a+b \end{pmatrix} \begin{pmatrix} a+b & \big| & C & D \\ M & \big| & a & b \end{pmatrix} \begin{pmatrix} a & b & \big| & N \\ C & D & \big| & a+b \end{pmatrix} \begin{pmatrix} a+b & \big| & U & P \\ N & \big| & a+b & 0 \end{pmatrix}$$

$$= \sum_{c} q^{2cp} \sum_{a+b=c} \sum_{M,N} \begin{Bmatrix} C & A & \big| & F \\ U & M & \big| & D \end{Bmatrix} \begin{Bmatrix} U & B & \big| & C \\ D & N & \big| & P \end{Bmatrix}$$

$$\begin{pmatrix} 0 & c & \big| & M \\ F & U & \big| & c \end{pmatrix} \begin{pmatrix} c & \big| & C & D \\ M & \big| & a & b \end{pmatrix} \begin{pmatrix} a & b & \big| & N \\ C & D & \big| & c \end{pmatrix} \begin{pmatrix} c & \big| & U & P \\ N & \big| & c & 0 \end{pmatrix}$$

$$= \sum_{c} q^{2cp} \sum_{M} \begin{Bmatrix} C & A & \big| & F \\ U & M & \big| & D \end{Bmatrix} \begin{Bmatrix} U & B & \big| & C \\ D & M & \big| & P \end{Bmatrix}$$

$$\begin{pmatrix} 0 & c & \big| & M \\ F & U & \big| & c \end{pmatrix} \begin{pmatrix} c & \big| & U & P \\ M & \big| & c & 0 \end{pmatrix} Y(M,C,D)Y(c,M),$$

the last step by (116). Therefore by (120) we have:

$$\sum_{E} \begin{Bmatrix} A & E & \big| & C \\ B & U & \big| & D \end{Bmatrix} \Lambda_{FE}^{AC} \Lambda_{EP}^{BD}$$

$$= \sum_{c} q^{2cp} \sum_{M} \begin{Bmatrix} C & A & \big| & F \\ U & M & \big| & D \end{Bmatrix} \begin{Bmatrix} U & B & \big| & C \\ D & M & \big| & P \end{Bmatrix} \begin{pmatrix} 0 & c & \big| & M \\ F & U & \big| & c \end{pmatrix} \begin{pmatrix} c & \big| & U & P \\ M & \big| & c & 0 \end{pmatrix}.$$

On the other hand we have:

$$
\begin{aligned}
&\sum_K \left\{\begin{array}{cc|c} F & A & C \\ B & U & K \end{array}\right\} \left\{\begin{array}{cc|c} A & B & K \\ P & U & D \end{array}\right\} \Lambda^{KU}_{FP} \\
&= \sum_K \left\{\begin{array}{cc|c} F & A & C \\ B & U & K \end{array}\right\} \left\{\begin{array}{cc|c} A & B & K \\ P & U & D \end{array}\right\} \sum_c q^{2pc} \left(\begin{array}{c|cc} 0 & U & K \\ F & c & -c \end{array}\right) \left(\begin{array}{cc|c} -c & c & P \\ K & U & 0 \end{array}\right) \\
&\quad = \sum_{K,M} \left\{\begin{array}{cc|c} F & A & C \\ B & U & K \end{array}\right\} \left\{\begin{array}{cc|c} A & B & K \\ P & U & D \end{array}\right\} \left\{\begin{array}{cc|c} U & K & F \\ U & M & P \end{array}\right\} \sum_c q^{2pc} \\
&\qquad\qquad \left(\begin{array}{cc|c} 0 & c & M \\ F & U & c \end{array}\right) \left(\begin{array}{c|cc} c & U & P \\ M & c & 0 \end{array}\right).
\end{aligned} \tag{155}
$$

The last step follow from by (122), (110) and (111). Now, from (124) and (126) we have:

$$
\begin{aligned}
&\sum_K \left\{\begin{array}{cc|c} F & A & C \\ B & U & K \end{array}\right\} \left\{\begin{array}{cc|c} A & B & K \\ P & U & D \end{array}\right\} \left\{\begin{array}{cc|c} U & K & F \\ U & M & P \end{array}\right\} \\
&\qquad = \sum_K \left\{\begin{array}{cc|c} A & B & K \\ P & U & D \end{array}\right\} \left\{\begin{array}{cc|c} A & B & K \\ U & F & C \end{array}\right\} \left\{\begin{array}{cc|c} U & K & F \\ U & M & P \end{array}\right\} \\
&\qquad = \left\{\begin{array}{cc|c} U & B & C \\ D & M & P \end{array}\right\} \left\{\begin{array}{cc|c} A & C & F \\ M & U & D \end{array}\right\} \\
&\qquad = \left\{\begin{array}{cc|c} C & A & F \\ U & M & D \end{array}\right\} \left\{\begin{array}{cc|c} U & B & C \\ D & M & P \end{array}\right\},
\end{aligned}
$$

Which finishes the proof.

Definition 108 (Balanced Representations of the Quantum Lorentz Group) *We have for any $p \in \mathbb{C}$ a representation $\rho(p)$ of the quantum Lorentz group in $V(p)$.*

These representations are called balanced representations of the Quantum Lorentz Group[17]

In [P], the reader can find the exact expression of the representations $\rho(p)$ in terms of the generators of the Quantum Lorentz Group, at least the unitary ones.

7.3.4 Simpler formulae for the Λ-coefficients in particular cases

Let $p \in \mathbb{C}$. We thus have an infinite dimensional representation $\rho(p)$ of the quantum Lorentz group given by the constants $\Lambda_{AD}^{BC}(p)$. Its representation space is by definition $V = \bigoplus_{\alpha\in\mathbb{N}_0} \overset{\alpha}{V}$. From (107), we can calculate the coefficients Λ_{AD}^{BC} if $B = 1/2$. In fact

Lemma 109 *Let $C \geq 0$ be an integer . We have*

$$\Lambda_{CC}^{1/2C-1/2}(p) = \frac{q^C\left(q^p + q^{-p}\right)}{q^{2C}+1}, \tag{156}$$

$$\Lambda_{CC}^{1/2C+1/2}(p) = -\frac{q^{C+1}\left(q^p + q^{-p}\right)}{q^{2C+2}+1}, \tag{157}$$

$$\Lambda_{CC+1}^{1/2C-1/2}(p) = \frac{q^{2C+2}q^p - q^{-p}}{q^{2C+2}+1}, \tag{158}$$

$$\Lambda_{C+1C}^{1/2C+1/2}(p) = \frac{q^{2C+2}q^{-p} - q^p}{q^{2C+2}+1} \tag{159}$$

Notice that all the other $\Lambda_{AD}^{BC}(p)$ coefficients with $B = 1/2$ are zero.

This also corrects equation that appears during the course of proof of theorem 3 of [BR2]. With this expression for the Λ-coefficients we can describe the action of the generating set $\{q^{J_z}, J_+, J_-, a, b, c, d\}$ of $\mathrm{Pol}(SU_q(2))$. From which follows:

[17]Recall that, in the classical case, this nomenclature is due to Barrett and Crane. See [BC]

Theorem 110 *Suppose $q = \exp(h/2)$. The representations $\rho(p)$ and $r(p + 4\pi ih)$ are equivalent.*

Let us prove the formula for $\Lambda^{1/2C-1/2}_{CC-1}(p)$ since the main results we are going to prove depend on it, ultimately. We have:

$$\begin{pmatrix} 1/2 & -1/2 & \bigg| & C+1 \\ C+1/2 & 1/2 & \bigg| & 0 \end{pmatrix} \begin{pmatrix} -1/2 & 1/2 & \bigg| & C \\ 1/2 & C+1/2 & \bigg| & 0 \end{pmatrix} = q^{-(C+1)^2}$$
$$\sqrt{\frac{[2C+3][2C+1][0]![1]![C]![1]![1]![C]![C+1]![C]![C+1]![C]!}{[1]![2C]![2C+1]![0]![2C+3]![2C+2]![C+1]![1]![0]![C+1]!}}$$
$$\sum_{V,W} \frac{q^{V(C+2)}q^{W(C+1)}(-1)^{V+C}(-1)^{W+1}[C+1+V]![W]![C+1-V]![2C+1-W]!}{[V]![W]![C+1-V]![C-W]![C-V]![1-W]![-C+V]![W]!}.$$

This last sum is to be extended to all V's and W's that make the terms inside the q-factorials positive, thus $V = C$ and $W = 0$ or $W = 1$. This last expression thus simplifies to:

$$\begin{aligned} q^{-(C+1)^2}\frac{[C]!^2}{[2C+2]![2C]!}&\frac{q^{C(C+2)}[2C+1]!}{[C]!}\left(-\frac{[2C+1]!}{[C]!} + q^{C+1}\frac{[2C]!}{[C-1]!}\right) \\ &= q^{-1}\frac{1}{[2C+2]}\left(-[2C+1] + q^{C+1}[C]\right) \\ &= \frac{1}{[2C+2](q-q^{-1})}\left(q^{-(2C+2)} - 1)\right) \\ &= -\frac{q^{-2C-2}}{1+q^{-2C-2}} = -\frac{1}{q^{2C+2}+1}. \end{aligned}$$

Whereas:

$$\begin{pmatrix} -1/2 & 1/2 & \vline & C+1 \\ C+1/2 & 1/2 & \vline & 0 \end{pmatrix} \begin{pmatrix} 1/2 & -1/2 & \vline & C \\ 1/2 & C+1/2 & \vline & 0 \end{pmatrix} = q^{(C+1)^2}$$

$$\sqrt{\frac{[2C+3][2C+1][0]![1]![C+1]![1]![0]![C+1]![C+1]![C]![C+1]![C]!}{[1]![2C]![2C+1]![0]![2C+3]![2C+2]![C]![1]![0]![C]!}}$$

$$\sum_{V,W} \frac{q^{V(C+2)}q^{W(C+1)}(-1)^{V+C+1}(-1)^W[C+V]![1+W]![C+2-V]![2C-W]!}{[V]![W]![C+1-V]![C-W]![C+1-V]![-W]![-C-1+V]![1+W]!},$$

thus $W=0$ and $V=C+1$. Therefore this last expression simplifies as:

$$q^{-(C+1)^2}\frac{[C+1]!^2}{[2C+2]![2C]!}q^{(C+1)(C+2)}\frac{[2C+1]![2C]!}{[C+1]![C]!} = q^{C+1}\frac{[C+1]}{[2C+2]} = \frac{q^{2C+2}}{1+q^{2C+2}}.$$

Notice a strange fact which can be seen from the just obtained expressions for the Λ-coefficients: The off diagonal terms are zero if $q=1$. This is actually immediate from (154). Comparing with 2.1.1, it appears that the representations $\rho(p)$ do not have the right limit. A solution for the mystery is obvious from equation (140).

7.3.5 Reapearence of finite dimensional representations

This material seems to be new. Recall the balanced representations $\rho(p), p \in \mathbb{C}$ of the Quantum Lorentz defined by (151), (152) and (154). This general form of representations of the Quantum Lorentz Group contains some finite dimensional representations. In fact, we have this consequence of Lemma 109:

Proposition 111 *Suppose $p \in \mathbb{N}$ then $\rho(p)$ has a finite dimensional subrepresentation $\rho(p)_{\text{fin}}$ in $V(p)_{\text{fin}} = {}^{0}_{V} \oplus ... \oplus {}^{C}_{V}$ where $C = p-1$.*

Compare with 2.2.1.

Proof. The elements a, b, c, d of $\mathrm{Pol}(SU_q(2))$ generate it, thus by (136) and the previous lemma keep $V(p)_{\mathrm{fin}}$ stable. ■

Definition 112 *The representations $\rho(p)_{\mathrm{fin}}, p \in \mathbb{N}$ are called finite dimensional balanced representations of the Quantum Lorentz Group.*

We can relate these representations with the ones defined in 7.2.4. First of all:

Proposition 113 *For any $p \in \mathbb{N}$ the representation $\rho(p)_{\mathrm{fin}}$ is irreducible.*

Proof. Compare with [BR2], proof of irreducibility in theorem 3. Let $f : V(p)_{\mathrm{fin}} \to V(p)_{\mathrm{fin}}$ be a $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ intertwiner. Since $\overset{\alpha}{\rho}$ is the irreducible representation of $U_q(\mathfrak{su}(2))$, the map f needs to send each $\overset{\alpha}{V}$ to itself through a multiple c_α of the identity, for $a = 0, ..., p-1$. Now

$$\overset{\frac{1}{2}\,-1/2}{g}_{-1/2}\overset{\alpha}{v}_{i_\alpha} = \sum_{\gamma, i_\gamma, D} \overset{\gamma}{v}_{i_\gamma} \left(\begin{array}{cc|c} i_\gamma & -1/2 & D \\ \gamma & 1/2 & x \end{array}\right) \left(\begin{array}{c|cc} x & 1/2 & \alpha \\ D & -1/2 & i_\alpha \end{array}\right) \Lambda^{1/2 D}_{\gamma\alpha}\overset{\gamma}{v}_{i_\gamma},$$

thus

$$\left\langle \overset{\alpha+1\,0}{v}, f\left(\overset{\frac{1}{2}\,-1/2}{g}_{-1/2}\overset{\alpha}{v}_0\right)\right\rangle =$$

$$c_{a+1} \left(\begin{array}{cc|c} 0 & -1/2 & \alpha+1/2 \\ \alpha+1 & 1/2 & -1/2 \end{array}\right) \left(\begin{array}{c|cc} -1/2 & 1/2 & \alpha \\ \alpha+1/2 & -1/2 & 0 \end{array}\right) \Lambda^{1/2\,\alpha+1/2}_{\alpha+1\,\alpha},$$

by (110) and (111). Analogously

$$\left\langle \overset{\alpha+1\,0}{v}, \overset{\frac{1}{2}\,-1/2}{g}_{-1/2} f(\overset{\alpha}{v}_0)\right\rangle =$$

$$c_{a} \left(\begin{array}{cc|c} 0 & -1/2 & \alpha+1/2 \\ \alpha+1 & 1/2 & -1/2 \end{array}\right) \left(\begin{array}{c|cc} -1/2 & 1/2 & \alpha \\ \alpha+1/2 & -1/2 & 0 \end{array}\right) \Lambda^{1/2\,\alpha+1/2}_{\alpha+1\,\alpha}.$$

Using (107), one sees that the Clebsch-Gordan coefficients in the last two formulae are not zero if $q \in (0,1)$. Indeed, since q is not a root of unity $[n]_q! \neq 0$ for any $n \in \mathbb{N}$, thus we just need to analyse the terms under the sum in (107). This term for the coefficient $\left(\begin{array}{cc|c} 0 & -1/2 & \alpha+1/2 \\ \alpha+1 & 1/2 & -1/2 \end{array}\right)$ simplifies as:

$$\begin{aligned}\sum_V q^{V(C+1)}(-1)^V & \frac{[C+1+V]![C+1-V]!]}{[V]![C+1-V]![C+1-V]![-C+V]!} \\ &= q^{(C+1)C}(-1)^C\left(\frac{[2C+1]!}{[C]!} - \frac{[2C+2]!}{[C+1]!}q^{C+1}\right) \\ &= q^C\frac{[2C+1]!}{[C]!}q^{2C+2}.\end{aligned}$$

Notice in the first sum we must have $V = C$ or $V = C+1$. The term under the sum of $\left(\begin{array}{c|cc} -1/2 & 1/2 & \alpha \\ \alpha+1/2 & -1/2 & 0 \end{array}\right)$ in (107) simplifies as:

$$\sum_V q^{V(C+1)}(-1)^V\frac{[V]![2C+1-V]!}{[V]![C+1-V]![1-V]![-1+V]!} = -q^{C+1}\frac{[2C]!}{[C]!}.$$

None of these terms are zero for $q \in (0,1)$. Thus from the explicit formulae for $\Lambda^{1/2\alpha+1/2}_{\alpha+1\alpha}$, one proves $c_a = c_{a+1}$ for $a = 0, ..., p-2$. ■

The representations $\rho(p)_{\text{fin}}$ are physical representations for any $p \in \mathbb{N}$, cf. Definition 104. Therefore Theorem 106 can be used. It tells us that either $\rho(p) = (\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi$ or $\rho(p) = (\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau$, where $a = \frac{p-1}{2}$. Let us see which one of these $\rho(p)$ is.

Let $f : V_{\text{fin}}(p) \to \overset{\alpha}{V} \otimes \overset{\alpha}{V}$ be an equivalence of representations of the Quantum Lorentz Group. The representation $\rho(p)_{\text{fin}}$ acts on the vector space $V^p_{\text{fin}} = \overset{0}{V} \oplus ... \oplus \overset{p-1}{V}$, and $U_q(\mathfrak{su}(2)) \subset U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ acts in it as $\overset{0}{\rho} \oplus ... \oplus \overset{p-1}{\rho}$. When restricted to $U_q(\mathfrak{su}(2))$ the morphism $\phi : U_q(\mathfrak{su}(2)) \to U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$ is simply the comultiplication in $U_q(\mathfrak{su}(2))$, and similarly with $\phi \circ \tau$. Therefore, as representations of $U_q(\mathfrak{su}(2))$, both $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi$ and $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau$ split as $\overset{0}{\rho} \oplus ... \oplus \overset{p-1}{\rho}$. Choose an identification $\overset{\alpha}{V} \otimes \overset{\alpha}{V} \cong \overset{0}{V} \oplus ... \oplus \overset{p-1}{V}$ given by the Clebsch-Gordan maps and the phases we have chosen from them. There must exist constants c_β such that

$$f\big|_{\overset{\beta}{V}} = c_\beta \mathrm{id}\big|_{\overset{\beta}{V}}, \beta = 0, ..., p$$

These constants are not zero since f is an isomorphism. Recall that $a = \overset{\frac{1}{2} -\frac{1}{2}}{g}_{-\frac{1}{2}}$, see 7.1.4. Therefore from (152), (108) and (109)

$$\left\langle \overset{0}{v}^0, a\overset{0}{v}_0 \right\rangle = \Lambda_{00}^{\frac{1}{2}\frac{1}{2}}. \tag{160}$$

Thus

$$\left\langle \overset{0}{v}^0, f(a\overset{0}{v}_0) \right\rangle = c_0 \Lambda_{00}^{\frac{1}{2}\frac{1}{2}} = -c_0 \frac{q^p + q^{-p}}{q + q^{-1}}$$

On the other hand we know $\phi(a) = q^{J_z} \otimes q^{-J_z}$. Therefore by definition of the representation $\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}$ and Clebsch-Gordan Coefficients we have:

$$\begin{aligned}
\left\langle \overset{0}{v}^0, \phi(a)\overset{0}{v}_0 \right\rangle &= \left\langle \overset{0}{v}^0, \sum_i \begin{pmatrix} 0 & \Big| & \alpha & \alpha \\ 0 & \Big| & i & -i \end{pmatrix} \left(q^{J_z} \otimes q^{-J_z}\right) \overset{\alpha}{v}_i \otimes \overset{\alpha}{v}_{-i} \begin{pmatrix} i & -i & \Big| & 0 \\ \alpha & a & \Big| & 0 \end{pmatrix} \right\rangle \\
&= \sum_{-\alpha \le i \le \alpha} \begin{pmatrix} 0 & \Big| & \alpha & \alpha \\ 0 & \Big| & i & -i \end{pmatrix} \begin{pmatrix} i & -i & \Big| & 0 \\ \alpha & a & \Big| & 0 \end{pmatrix} q^{2i} \\
&= \sum_{-\alpha \le i \le \alpha} \frac{1}{d_a} (-1)^{2\alpha} (-1)^{-2i} q^{4i}, \text{ by (118) and (112)} \\
&= \frac{q^p + q^{-p}}{q + q^{-1}}
\end{aligned}$$

For the last step, notice 2α and $2i$ must have the same parity. Therefore

$$\left\langle \overset{0}{v}^0, \phi(a) f(\overset{0}{v}_0) \right\rangle = c_0 \frac{q^p + q^{-p}}{q + q^{-1}}$$

Since $c_0 \neq 0$, comparing with (101) we conclude we must have $\rho(p)_{\text{fin}} = (\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau$, since $\tau(a) = -a$. We have proved:

Theorem 114 *Let $p \in \mathbb{N}$. The representation $\rho(p)_{\text{fin}}$ is equivalent to $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau$, where $\alpha = \frac{p-1}{2}$.*

This re-appearence of finite dimensional representations is also obvious from the formulae for the unitary representations of the quantum Lorentz Group in [P].

7.3.6 Unitary Balanced Representations

Up to now we have not studied the unitarity of the representations of the Quantum Lorentz Group. Let us now define a class of unitary representations of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$, the unitary balanced representations. They are called simple in [NR].

This discussion is very similar to the one in [BR2], and only is it slightly different because we use a different definition of Λ^{BC}_{AD}. Consider $V = \bigoplus_{\alpha \in \mathbb{N}_0} \overset{\alpha}{V}$. It has an inner product such that $\{\overset{\alpha}{v}_{i_\alpha}, \alpha \in \mathbb{N}_0, -\alpha \leq i_\alpha \leq \alpha\}$, since the spin α representation $\overset{\alpha}{\rho}$ of $U_q(\mathfrak{su}(2))$ are unitary, we automatically have $< \rho(a)v, w >=< v, \rho(a^*)v >$ if $v \in V$ and $a \in U_q(\mathfrak{su}(2))$. Let us find out the conditions so that ρ is also a unitary representation of $\text{Pol}(SU_q(2))$, thus also of $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$. Recall $\overset{\alpha}{g}{}^{i_\alpha *}_{j_\alpha} = q^{j_\alpha - i_\alpha}(-1)^{i_\alpha - j_\alpha} \overset{\alpha}{g}{}^{-i_\alpha}_{-j_\alpha}$. We have:

$$\left\langle \overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha} \overset{\beta}{v}_{i_\beta}, \overset{\gamma}{v}_{i_\gamma} \right\rangle = \frac{\mathcal{F}_\beta}{\mathcal{F}_\gamma} \sum_M \sum_{-M \leq y \leq M} \begin{pmatrix} i_\gamma & i_\alpha & \big| & M \\ \gamma & \alpha & \big| & y \end{pmatrix} \begin{pmatrix} y & \big| & \alpha & \beta \\ M & \big| & j_\alpha & i_\beta \end{pmatrix} \Lambda^{\alpha M}_{\gamma\beta}$$

whereas:

$$\left\langle \overset{\beta}{v}_{i_\beta}, \overset{\alpha i_\alpha}{g}_{j_\alpha}{}^{*} \overset{\gamma}{v}_{i_\gamma} \right\rangle$$

$$= \frac{\mathcal{F}_\gamma}{\mathcal{F}_\beta} q^{j_\alpha - i_\alpha} (-1)^{i_\alpha - j_\alpha} \overset{\alpha - i_\alpha}{g}_{-j_\alpha} \sum_D \sum_{-D \le x \le D} \left(\begin{array}{cc|c} i_\beta & -i_\alpha & D \\ \beta & \alpha & x \end{array} \right) \left(\begin{array}{c|cc} x & \alpha & \gamma \\ D & -j_\alpha & i_\gamma \end{array} \right) \overline{\Lambda^{\alpha D}_{\beta\gamma}}$$

where the bar denotes complex conjugation. Notice q and the Clebsch-Gordan coefficients are real. The last expression can be written as

$$\frac{\mathcal{F}_\gamma}{\mathcal{F}_\beta} \sum_M \sum_{-M \le y \le M} \left(\begin{array}{cc|c} i_\gamma & i_\alpha & M \\ \gamma & \alpha & y \end{array} \right) \left(\begin{array}{c|cc} y & \alpha & \beta \\ M & j_\alpha & i_\beta \end{array} \right) \overline{\Lambda^{\alpha D}_{\gamma\beta}}$$

Using,(114), (122) and (123). Therefore unitarity holds if we have :

$$\frac{\mathcal{F}_\gamma}{\mathcal{F}_\beta} \overline{\Lambda^{\alpha D}_{\gamma\beta}}(p) = \frac{\mathcal{F}_\beta}{\mathcal{F}_\gamma} \Lambda^{\alpha D}_{\gamma\beta}(p)$$

Because a, b, c, d generate $\mathrm{Pol}(SU_q(2))$ we only need to prove this if $\alpha = \frac{1}{2}$. By lemma 109, this is ensured if $p \in i\mathbb{R}$, for we can find a recursive expression for $\mathcal{F}_a$. Explicit formulae for these constant appear in [BR2]. This last paragraph almost paraphrases [BR2]

Definition 115 *Let $p \in \mathbb{R}$, the unitary balanced representation of $U_q(\mathfrak{sl}(2,\mathbb{C})_\mathbb{R})$ with parameter p is the unitary representation $\rho(p)$ just constructed.*

These representations are the quantised counterpart of the balanced representations of the Lorentz group defined in 2.1.1 . They are irreducible in the sense any quantum Lorentz group intertwiner $V(p) \to V(p)$ must be a multiple of the identity. The same argument used to prove the irreducibility of the finite dimensional representations applies to prove this. This is a result of [BR2].

7.3.7 Formal R-matrix and Group Like elements on the Quantum lorentz Group

We continue to follow [BR2] closely. Let $\mathcal{A}$ be a finite dimensional Hopf algebra and let $\{v_i\}$ be a basis of $\mathcal{A}$. Let also $\{v^i\}$ be the dual basis of $\mathcal{A}^*$. The quantum double of $\mathcal{A}$ has an R-matrix given by $R = \sum_i v_i \otimes 1 \otimes 1 \otimes v^i \in \mathcal{D}(\mathcal{A}, \mathcal{A}^{*\mathrm{cop}}) \otimes \mathcal{D}(A, \mathcal{A}^{*\mathrm{cop}})$. If $\mathcal{A}$ is infinite dimensional, then the expression for R is an infinite sum, however, it may happen that it makes sense when applied to the tensor product of representations of $\mathcal{A}$. This is what happens in the quantum Lorentz group case for the balanced representations $\rho(p), p \in \mathbb{C}$ and $\rho(p)_{\mathrm{fin}}, p \in \mathbb{N}$.

Consider the basis $\{\overset{\alpha i_\alpha}{g}_{j_\alpha}\}$ of $\mathrm{Pol}(SU_q(2))$, where $\alpha \in \frac{1}{2}\mathbb{N}_0$ and $-\alpha \leq i_\alpha, j_\alpha \leq \alpha$. When trying to define the dual basis of it in $U_q(\mathfrak{su}(2))$, we are led to find elements $\overset{\alpha i_\alpha}{z}_{j_\alpha} \in U_q(\mathfrak{su}(2))$ such that

$$\left\langle \overset{\beta i_\beta}{v} | \overset{\beta}{\rho}(\overset{\alpha i_\alpha}{z}_{j_\alpha}) | \overset{\beta}{v}_{j_\beta} \right\rangle = \left\langle \overset{\beta i_\beta}{g}_{j_\beta}, \overset{\alpha i_\alpha}{z}_{j_\alpha} \right\rangle = \delta(\alpha, \beta)\delta(i_\alpha, j_\beta)\delta(j_\alpha, i_\beta),$$

whose existence is dubious. The best solution is to modify $U_q(\mathfrak{su}(2))$ slightly. For a vector space V let $L(V)$ be the algebra of linear maps $V \to V$. Define

$$\hat{U}_q(\mathfrak{su}(2)) = \bigoplus_{\alpha \in \frac{1}{2}\mathbb{N}_0} L(\overset{\alpha}{V}).$$

Recall that $U_q(\mathfrak{su}(2))$ is embedded in $\prod_{\alpha \in \frac{1}{2}\mathbb{N}_0} L(\overset{\alpha}{V})$ through $x \mapsto \prod_{\alpha \in \frac{1}{2}\mathbb{N}_0} \overset{\alpha}{\rho}(x)$. This morphism is injective for the elements of $U_q(\mathfrak{su}(2))$ are separated by the finite dimensional representations of it. Therefore $U_q(\mathfrak{su}(2))$ is *almost* a subalgebra $\hat{U}_q(\mathfrak{su}(2))$, in fact it is embedded in its subalgebra of multipliers, cf [BR2]. Let $\overset{\alpha i_\alpha}{X}_{j_\alpha}, \alpha \in \frac{1}{2}\mathbb{N}_0, -\alpha \leq i_\alpha, j_\alpha \leq \alpha$ be the elements of $\hat{U}_q(\mathfrak{su}(2))$ such that $\overset{\alpha i_\alpha}{X}_{j_\alpha} \overset{\beta}{v}_{i_\beta} = \delta(\alpha, \beta)\delta(i_\alpha, i_\beta)\overset{\alpha}{v}_{j_\alpha}$. In particular we have:

$$\overset{\alpha i_\alpha}{X}_{j_\alpha} \overset{\beta i_\beta}{X}_{j_\beta} = \overset{\alpha i_\beta}{X}_{j_\alpha}, \delta(\alpha, \beta)\delta(i_\alpha, j_\beta). \tag{161}$$

There exists an obvious duality between $\hat{U}_q(\mathfrak{su}(2))$ and $\mathrm{Pol}(SU_q(2))$ which verifies

$$\left\langle \overset{\alpha i_\alpha}{X}_{j_\alpha}, \overset{\beta i_\beta}{g}_{j_\beta} \right\rangle = \delta(\alpha,\beta)\delta(i_\alpha, j_\beta)\delta(j_\alpha, i_\beta).$$

It *extends* the duality between $U_q(\mathfrak{su}(2))$ and $\mathrm{Pol}(SU_q(2))$, since

$$\left\langle \overset{\alpha i_\alpha}{X}_{j_\alpha}, \overset{\beta i_\beta}{g}_{j_\beta} \right\rangle = \left\langle \overset{\beta i_\beta}{v}, \overset{\alpha i_\alpha}{X}_{j_\alpha} \overset{\beta}{v}_{j_\beta} \right\rangle = \delta(\alpha,\beta)\delta(i_\alpha, j_\beta) \left\langle \overset{\beta i_\beta}{v}, \overset{\alpha}{v}_{j_\alpha} \right\rangle = \delta(\alpha,\beta)\delta(i_\alpha, j_\beta)\delta(i_\beta, j_\alpha).$$

We define a coproduct in $\hat{U}_q(\mathfrak{su}(2))$ by dualising the product of $\mathrm{Pol}(SU_q(2))$. Therefore

$$\Delta(\overset{\alpha i_\alpha}{X}_{j_\alpha}) = \sum_{\beta,\gamma,i_\beta,i_\gamma} \left(\begin{array}{cc|c} j_\beta & j_\gamma & \alpha \\ \beta & \gamma & j_\alpha \end{array} \right) \left(\begin{array}{c|cc} i_\alpha & \beta & \gamma \\ \alpha & i_\beta & i_\gamma \end{array} \right) \overset{\beta i_\beta}{X}_{j_\beta} \otimes \overset{\gamma i_\gamma}{X}_{j_\gamma}.$$

This infinite sum is interpreted as an element of $\prod_{\alpha,\beta} \overset{\alpha}{V} \otimes \overset{\beta}{V}$. The evaluations $(\mathrm{id} \otimes \Delta)(\overset{\alpha i_\alpha}{X}_{j_\alpha})$ and $(\Delta \otimes \mathrm{id})(\overset{\alpha i_\alpha}{X}_{j_\alpha})$ also make sense when interpreted as elements of $\prod_{\alpha,\beta,\gamma} \overset{\alpha}{V} \otimes \overset{\beta}{V} \otimes \overset{\gamma}{V}$. The coproduct Δ is coassociative. The algebra $\hat{U}_q(\mathfrak{su}(2))$ also has a star structure such that $(\overset{\alpha i_\alpha}{X}_{j_\alpha})^* = \overset{\alpha j_\alpha}{X}_{i_\alpha}$, thus it is simply matrix transposition. It *extends* the star structure in $U_q(\mathfrak{su}(2))$, given that the representations $\overset{\alpha}{\rho}$ of $U_q(\mathfrak{su}(2))$ in $\overset{\alpha}{V}$ are unitary.

The expression (137) defining the product quantum double also makes sense when considering $\mathcal{A} = \hat{U}_q(\mathfrak{su}(2))$ and $\mathcal{A}^* = \mathrm{Pol}(SU_q(2))$, see [BR2]. We therefore define $\hat{\mathcal{D}} = \mathcal{D}(\hat{U}_q(\mathfrak{su}(2)), \mathrm{Pol}(SU_q(2)))$, thus $\hat{\mathcal{D}}$ is an associative algebra. It also has a star structure such that

$$(\overset{\alpha i_\alpha}{X}_{j_\alpha} \otimes \overset{\beta i_\beta}{g}_{j_\beta})^* = (\overset{\alpha i_\alpha}{X}_{j_\alpha})^* \otimes (\overset{\beta i_\beta}{g}_{j_\beta})^* = \overset{\alpha j_\alpha}{X}_{i_\alpha} \otimes S(\overset{\beta i_\beta}{g}_{j_\beta}).$$

This last antipode is taken in $\mathrm{Pol}(SU_q(2))$.

As pointed out in [BR2], each balanced representation $\rho(p)$ of $\mathcal{D}$ gives rise to a balanced representation $\hat{\rho}(p)$ of $\hat{\mathcal{D}}$. This extension is quite natural. The representation space of $\rho(p)$ is $V = \bigoplus_{\alpha \in \mathbb{N}_0}$. Put $\overset{\alpha i_\alpha}{X}_{j_\alpha} = \overset{\alpha i_\alpha}{X}_{j_\alpha} \otimes 1$ and $\overset{\alpha i_\alpha}{g}_{j_\alpha} = 1 \otimes \overset{\alpha i_\alpha}{g}_{j_\alpha}$. The elements $\overset{\alpha i_\alpha}{g}_{j_\alpha}$ act in V as before, namely as in (152). Recall that if $a \in U_q(\mathfrak{su}(2))$ then $\rho(a) = \prod_{\alpha \in \mathbb{N}_0} \overset{\alpha}{\rho}(a)$ if $a \in U_q(\mathfrak{su}(2))$. It is thus natural to define

$$\rho(\overset{\alpha i_\alpha}{X}_{j_\alpha})\overset{\beta}{v}_{i_\beta} = \delta(\alpha, b)\delta(i_\alpha, i_\beta)\overset{\alpha}{v}_{j_\alpha}. \tag{162}$$

As proved in [BR2] this action does define a representation of $\hat{\mathcal{D}}$. This representation is still unitary for $p \in i\mathbb{R}$. For the case $p \in \mathbb{N}$ we still have a subrepresentation $\hat{\rho}(p)_{\text{fin}}$ of $\hat{\mathcal{D}}$. See 7.3.5

The formal R-matrix $\mathcal{R}$ in the Quantum Lorentz Group is the infinite sum

$$\mathcal{R} = \sum_{\alpha, i_\alpha, j_\alpha} \overset{\alpha i_\alpha}{X}_{j_\alpha} \otimes \overset{\alpha j_\alpha}{g}_{i_\alpha}. \tag{163}$$

It is easy to see how it acts in the tensor product of two balanced representations $r(p)$:

$$\mathcal{R}\overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta} = \sum_{D,\gamma,j_\alpha} \left(\begin{array}{cc|c} i_\gamma & j_\alpha & D \\ \gamma & \alpha & x \end{array}\right) \left(\begin{array}{c|cc} x & \alpha & \beta \\ D & i_\alpha & i_\beta \end{array}\right) \frac{\mathcal{F}_\beta}{\mathcal{F}_\gamma} \Lambda^{\alpha D \alpha}_{\gamma\beta} \overset{\alpha}{v}_{j_\alpha} \otimes \overset{\gamma}{v}_{i_\gamma}. \tag{164}$$

The action of $\mathcal{R}$ in $V \otimes V$ is thus well defined, even though $\mathcal{R}$ is defined by an infinite sum. This is a result of [BR1]. Notice we are considering the algebraic, rather than topological tensor product of vector spaces. Obviously, for the case $p \in \mathbb{N}$ the action of $\mathcal{R}$ in $\rho(p)_{\text{fin}} \otimes \rho(p)_{\text{fin}}$ truncates to the finite sum $\sum_{\alpha=0}^{p-1} \overset{\alpha i_\alpha}{X}_{j_\alpha} \otimes \overset{\alpha j_\alpha}{g}_{i_\alpha}$.

To determine $\mathcal{R}^{-1}$ we can use the identity $(\text{id} \otimes S^{-1})(R) = R^{-1}$ verified by any R-matrix in a Hopf algebra. We have $S_{\mathcal{D}} = S^{-1}$, where the last antipode is taken in $\text{Pol}(SU_q(2))$ and the first is the one in $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$, see 7.1.4 and 7.1.5. This leads to the following expression for $\mathcal{R}^{-1}$:

$$\mathcal{R}^{-1} = \sum_{\alpha, i_\alpha, j_\alpha} q^{-i_\alpha + j_\alpha}(-1)^{i_\alpha - j_\alpha} \overset{\alpha i_\alpha}{X}_{j_\alpha} \otimes \overset{\alpha - i_\alpha}{g}_{-j_\alpha}, \tag{165}$$

which makes sense when applied to the tensor product of two balanced representations $\rho(p)$.

Obviously, our formal calculations do not prove that $\mathcal{R}^{-1}$ is the inverse of $\mathcal{R}$ or that the action of $\mathcal{R}$ in the balanced representations of the Lorentz group defines a braid group representation. However, we can show this facts directly. For example the fact $\mathcal{R}\mathcal{R}^{-1} = \mathrm{id}$ (action in balanced representations) is a consequence of equation (133), together with the fact $\hat{\rho}(p)$ is a representation of $\hat{\mathcal{D}}$ if for any representation $\rho(p)$ of $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$. In fact:

$$
\begin{aligned}
\mathcal{R}\mathcal{R}^{-1} &= \sum_{\alpha,\beta,i_\alpha,i_\beta,j_\alpha,j_\beta} \left(\overset{\beta i_\beta}{X}_{j_\beta} \otimes \overset{\alpha j_\beta}{g}_{i_\beta}\right)\left(q^{-i_\alpha+j_\alpha}(-1)^{i_\alpha-j_\alpha}\overset{\alpha i_\alpha}{X}_{j_\alpha} \otimes \overset{\alpha -i_\alpha}{g}_{-j_\alpha}\right) \\
&= \sum_{a,j_\alpha,i_\alpha,j_\beta} q^{-i_\alpha+j_\alpha}(-1)^{2\alpha+i_\alpha-j_\alpha}\overset{\alpha i_\alpha}{X}_{j_\beta} \otimes \overset{\alpha i_\alpha}{g}_{j_\beta}\overset{\alpha -i_\alpha}{g}_{-j_\alpha} \\
&= \sum_{a,j_\alpha,i_\alpha,j_\beta,\gamma,i_\gamma,j_\gamma} \left(\begin{array}{cc|c} i_\alpha & -i_\alpha & \gamma \\ \alpha & \alpha & i_\gamma \end{array}\right)\left(\begin{array}{c|cc} j_\alpha & \alpha & \alpha \\ \gamma & j_\beta & -j_\alpha \end{array}\right) \\
&\qquad\qquad q^{-i_\alpha+j_\alpha}(-1)^{i_\alpha-j_\alpha}\overset{\alpha j_\alpha}{X}_{j_\beta} \otimes \overset{\gamma i_\gamma}{g}_{j_\gamma} \\
&= \sum_{\alpha,j_\alpha}\overset{\alpha i_\alpha}{X}_{i_\alpha} \otimes \overset{0 0}{g}_{0}
\end{aligned}
$$

this last one acts as the identity in the representations $\rho(p)$. The Yang-Baxter equation is a consequence of the identities $(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{1,3}\mathcal{R}_{1,2}$ and $(\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{1,3}\mathcal{R}_{1,2}$, when considering the action in the balanced represesentations, see [K] for example. These last equalities are tautologies when we look at the structure maps of $\hat{\mathcal{D}}$.

It is easy to verify that $\mathcal{R}^{*,*} = \mathcal{R}^{-1}$, by the very expression of R^{-1} and the star structure. Therefore the R-matrix which we consider in the quantum Lorentz Group is compactible with its star structure, see 7.1.1. Let $p \in i\mathbb{R}$, thus $\rho(p)$ is a unitary representation of $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ thus also $\hat{\rho}$ is a unitary representation of $\hat{\mathcal{D}}$. Therefore we have:

Theorem 116 *For any $p \in \mathbb{C}$ we have a braid group representation in $V(p)^{\otimes n}$ given by the formal R-matrix in the quantum Lorentz Group. Moreover if $p \in i\mathbb{R}$ then all the braidding operators will extend to unitary operators, uniquely.*

The quantum Lorentz group also has a group like element which is $G = q^{2J_z}$. As referred in [BNR] it formally gives $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ a structure of ribbon Hopf algebra. We shall see later that it is indeed a good group like element.

7.3.8 The action of the R-matrix and Group Like element in finite dimensional balanced representations

As we referred to in 7.2.4, if $\mathcal{A}$ is a Hopf algebra with an R-matrix R, then the Hopf algebra map $\phi : \mathcal{D}(\mathcal{A}, \mathcal{A}^{*\mathrm{cop}}) \to \mathcal{A}\hat{\otimes}\mathcal{A}$ preserves the R-matrices if $\mathcal{A}$ is finite dimensional. See [BR2] and [M]. Let us see that this is the case for the Quantum Lorentz Group. We need to modify the statement slightly, for all R-matrices are formal and we work with infinite dimensional representations. We freely use the notation of 7.2.4, 7.3.1 and 7.3.7.

The algebra $\mathrm{Pol}(SU_q(2))$ admits a $\mathbb{Z}_2$ grading $\mathrm{Pol}(SU_q(2)) = \mathrm{Pol}(SU_q(2))_{\mathrm{even}} \oplus \mathrm{Pol}(SU_q(2))_{\mathrm{odd}}$, where $\mathrm{Pol}(SU_q(2))_{\mathrm{even}}$ is generated by all the $\overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha}$ with $\alpha \in \mathbb{N}$ and $\mathrm{Pol}(SU_q(2))_{\mathrm{odd}}$ is generated by all the $\overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha}$ with $\alpha \in \mathbb{N} + \frac{1}{2}$. As referred to at the end of 7.1.4 there exists an automorphism of $\mathrm{Pol}(SU_q(2))$ which equals the identity in the even part of $\mathrm{Pol}(SU_q(2))$ and minus the identity in the odd part of $\mathrm{Pol}(SU_q(2))$. This automorphism extends to an automorphism of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ in a way such that it is the identity in $U_q(\mathfrak{su}(2)) \otimes 1$. As pointed out in 7.3.1 any irreducible physical representation of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ with minimal spin zero is either equivalent to $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi$ or to $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau$. This is a theorem of Takeuchi, see [T], answering affirmatively a conjecture formulated Woronowicz and Poddles, see [PW]. In Theorem 114, we used this result to prove that

$$\rho(p)_{\text{fin}} \cong (\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau, \alpha = (p-1)/2, p \in \mathbb{N}.$$

There exists a map $f : V(p)_{\text{fin}} \to \overset{\alpha}{V} \otimes \overset{\alpha}{V}$ intertwining $\rho(p)_{\text{fin}}$ and $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau$. From the form of the R-matrix $\mathcal{R}$ of the Quantum Lorentz Group, it is obvious that the action of $\mathcal{R}$ is the same in tensor products of representations of the form $\rho(p)_{\text{fin}}$ and of $\rho(p)_{\text{fin}} \circ \tau$. Notice again that we are only considering integer spins, thus we only need the "even part" of the formal R-matrix $\mathcal{R}$. We will use this fact later.

Lemma 117 *Let $f : \phi : V(p)_{\text{fin}} \to \overset{\alpha}{V} \otimes \overset{\alpha}{V}$ be an intertwiner, and $\hat{R}$ be the R-matrix of $U_q(\mathfrak{su}(2)) \hat{\otimes} U_q(\mathfrak{su}(2))$, we have:*

$$(f \otimes f)(\mathcal{R} x \otimes y) = \hat{R}(f \otimes f)(x \otimes y)$$

Let us prove this lemma. In the case $\mathcal{A}$ is finite dimensional, this is a consequence of the fact $(\phi \otimes \phi)(R) = \hat{R}$, where the first R-matrix is the R-matrix of the quantum double. This, as referred in 7.2.4, follows from the equalities $(\Delta \otimes \text{id})(R) = R_{13}R_{23}$ and $(\text{id} \otimes \Delta)(R) = R_{13}R_{23}$, satisfied by any R-matrix in a Hopf algebra. The discussion now is very similar to the one used in the proof of Theorem 114.

The morphism $\phi : \mathcal{D} \to U_q(\mathfrak{su}(2)) \hat{\otimes} U_q(\mathfrak{su}(2))$ *extends* to a morphism $\hat{\phi} : \hat{\mathcal{D}} \to \prod_{\alpha,\beta \in \frac{1}{2}\mathbb{N}_0} L(\overset{\alpha}{V}) \otimes L(\overset{\beta}{V})$. Let us prove that for any $v \in V(p)_{\text{fin}}$ we have

$$f\left(\overset{\omega i_\omega}{X}_{j_\omega} v\right) = \hat{\phi}\left(\overset{\omega i_\omega}{X}_{j_\omega}\right) f(v) \tag{166}$$

Notice that $\overset{\omega i_\omega}{X}_{j_\omega}$ is not in $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_\mathbb{R})$, thus this fact needs proving. First of all we need to define:

$$\hat{\phi}(\overset{\omega i_\omega}{X}_{j_\omega}) = \Delta(\overset{\omega i_\omega}{X}_{j_\omega}) = \sum_{\beta,\gamma,i_\beta,i_\gamma,j_\beta,j_\gamma} \left(\begin{array}{cc|c} j_\beta & j_\gamma & \omega \\ \beta & \gamma & j_\omega \end{array}\right) \left(\begin{array}{c|cc} i_\omega & \beta & \gamma \\ \omega & i_\beta & i_\gamma \end{array}\right) \overset{\beta i_\beta}{X}_{j_\beta} \otimes \overset{\gamma i_\gamma}{X}_{j_\gamma}$$

which acts in $\overset{\alpha}{V} \otimes \overset{\alpha}{V}$ in the obvious way. That is as

$$\hat{\phi}(\overset{\omega i_\omega}{X}_{j_\omega}) = \Delta(\overset{\omega i_\omega}{X}_{j_\omega}) = \sum_{i_\beta,i_\gamma,jb,jg} \left(\begin{array}{cc|c} j_\beta & j_\gamma & \omega \\ \alpha & \alpha & j_\omega \end{array}\right) \left(\begin{array}{c|cc} i_\omega & \alpha & \alpha \\ \omega & i_\beta & i_\gamma \end{array}\right) \overset{\alpha i_\beta}{X}_{j_\beta} \otimes \overset{\alpha i_\gamma}{X}_{j_\gamma}$$

Recall that $\phi \circ \tau$ when restricted to $U_q(\mathfrak{su}(2))$ is the comultiplication in it. Therefore, as a representation of $U_q(\mathfrak{su}(2))$ the representation $(\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}) \circ \phi \circ \tau$ of $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_\mathbb{R})$ splits as $\overset{0}{\rho} \oplus ... \oplus \overset{p-1}{\rho}$. Fix once for all the identification $\overset{\alpha}{V} \otimes \overset{\alpha}{V} \cong \overset{0}{V} \oplus ... \oplus \overset{p-1}{V}$ given by the Clebsch-Gordan maps $\Psi^{\alpha\alpha}_\beta$ and $\Phi^\beta_{\alpha\alpha}$ in the phases we have chosen for them. Under this identification $\hat{\phi}(\overset{\omega i_\omega}{X}_{j_\omega})$ acts in $\overset{\alpha}{V} \otimes \overset{\alpha}{V}$ as

$$\hat{\phi}(\overset{\omega i_\omega}{X}_{j_\omega})\overset{\beta}{v}_i = \delta(\omega,\beta)\delta(i_\omega, i)\overset{\beta}{v}_{j_\omega} \tag{167}$$

Indeed, we have:

$$\begin{aligned}
&\hat{\phi}(\overset{\omega i_\omega}{X}_{j_\omega})\overset{\beta}{v}_i \\
&= \sum_\varepsilon \Psi^{\alpha\alpha}_\varepsilon \left(\sum_{i_\beta,i_\gamma,j_\beta,j_\gamma} \left(\begin{array}{cc|c} j_\beta & j_\gamma & \omega \\ \alpha & \alpha & j_\omega \end{array}\right) \left(\begin{array}{c|cc} i_\omega & \alpha & \alpha \\ \omega & i_\beta & i_\gamma \end{array}\right) \overset{\alpha i_\beta}{X}_{j_\beta} \otimes \overset{\alpha i_\gamma}{X}_{j_\gamma} \right) \Phi^\beta_{\alpha\alpha}(\overset{\beta}{v}_i) \\
&= \sum_{\varepsilon,i_\varepsilon,i_\beta,i_\gamma,j_\beta,j_\gamma} \left(\begin{array}{c|cc} i_\varepsilon & \alpha & \alpha \\ \varepsilon & j_\beta & j_\gamma \end{array}\right) \left(\begin{array}{cc|c} j_\beta & j_\gamma & \omega \\ \alpha & \alpha & j_\omega \end{array}\right) \left(\begin{array}{c|cc} i_\omega & \alpha & \alpha \\ \omega & i_\beta & i_\gamma \end{array}\right) \left(\begin{array}{cc|c} i_\beta & i_\gamma & \beta \\ \alpha & \alpha & i \end{array}\right) \overset{\varepsilon}{v}_{i_\varepsilon} \\
&= \delta(i_\omega, i)\delta(\omega,\beta)\overset{\beta}{v}_{j_\omega}
\end{aligned}$$

The last equality from (116). This proves that equation (167) is correct.

We have two representation of $U_q(\mathfrak{su}(2))$, one in $V(p)_{\text{fin}}$ and the other in $\overset{\alpha}{V}\otimes\overset{\beta}{V}$. Since f is (in particular) a $U_q(\mathfrak{su}(2))$-intertwiner there must exist $c_\beta\in\mathbb{C}$ such that $f(\overset{\beta}{v}_{i_\beta})=c_\beta\overset{\beta}{v}_{i_\beta}$. In particular we have:

$$f\left(\overset{\omega i_\omega}{X}_{j_\omega}\overset{\beta}{v}_{i_\beta}\right)=\delta(i_\omega,i_\beta)\delta(\omega,\beta)f(\overset{\beta}{v}_{j_\beta})=\delta(i_\omega,i_\beta)\delta(\omega,\beta)c_\beta\overset{\beta}{v}_{j_\beta}=\hat{\phi}(\overset{\omega i_\omega}{X}_{j_\omega})f(\overset{\beta}{v}_{i_\beta}),$$

which proves (166), is correct. Therefore, if $\beta,\gamma\in\{0,1,...,p-1\}$, we have

$$\begin{aligned}
&(f\otimes f)(\mathcal{R}\overset{\beta}{v}_{i_\beta}\otimes\overset{\gamma}{v}_{i_\gamma})\\
&\qquad=(f\otimes f)\left(\sum_{a\in\frac{1}{2}\mathbb{N}_0}(\overset{\alpha i_\alpha}{X}_{j_\alpha}\otimes\overset{\alpha j_\alpha}{g}_{i_\alpha})\left(\overset{\beta}{v}_{i_\beta}\otimes\overset{\gamma}{v}_{i_\gamma}\right)\right)\\
&\qquad=(f\otimes f)\left(\sum_{a\in\mathbb{N}}(\overset{\alpha i_\alpha}{X}_{j_\alpha}\otimes\overset{\alpha j_\alpha}{g}_{i_\alpha})\left(\overset{\beta}{v}_{i_\beta}\otimes\overset{\gamma}{v}_{i_\gamma}\right)\right)\\
&\qquad=(f\otimes f)\left(\sum_{a\in\mathbb{N}}(-1)^{2\alpha}(\overset{\alpha i_\alpha}{X}_{j_\alpha}\otimes\overset{\alpha j_\alpha}{g}_{i_\alpha})\left(\overset{\beta}{v}_{i_\beta}\otimes\overset{\gamma}{v}_{i_\gamma}\right)\right)\\
&\qquad=(f\otimes f)\left(\sum_{a\in\mathbb{N}}(\overset{\alpha i_\alpha}{X}_{j_\alpha}\otimes\tau(\overset{\alpha j_\alpha}{g}_{i_\alpha}))\left(\overset{\beta}{v}_{i_\beta}\otimes\overset{\gamma}{v}_{i_\gamma}\right)\right)\\
&\qquad=\sum_{\alpha\in\mathbb{N}}(\hat{\phi}\otimes\phi)(\overset{\alpha i_\alpha}{X}_{j_\alpha}\otimes\overset{\alpha j_\alpha}{g}_{i_\alpha})\left(f(\overset{\beta}{v}_{i_\beta})\otimes f(\overset{\gamma}{v}_{i_\gamma})\right)\\
&\qquad=\sum_{\alpha\in\mathbb{N}}\sum_{\left(\overset{\alpha j_\alpha}{g_{i_\alpha}}\right)}\Delta(\overset{\alpha i_\alpha}{X}_{j_\alpha})\otimes(\overset{\alpha j_\alpha}{g}_{i_\alpha}{}''\otimes\mathrm{id})(R^{(+)})\otimes(\overset{\alpha j_\alpha}{g}_{i_\alpha}{}'\otimes\mathrm{id})(R^{(-)})\left(f(\overset{\beta}{v}_{i_\beta})\otimes f(\overset{\gamma}{v}_{i_\gamma})\right)
\end{aligned}$$

Thus, similarly to in the finite dimensional case, all follows from $(\Delta\otimes\mathrm{id})(R)=R_{13}R_{23}$ and $(\mathrm{id}\otimes\Delta)(R)=R_{13}R_{23}$. Indeed, after 7.2.2 the calculations in 7.2.3 are valid when considering the action in finite dimensional representations. Let us see it is so. Pick up a $\beta,\gamma,\delta,\varepsilon\in\frac{1}{2}\mathbb{N}_0$. Let use see how the term

$$\sum_{\alpha\in\mathbb{N}}\sum_{\left(\overset{\alpha j_\alpha}{g_{i_\alpha}}\right)}\Delta(\overset{\alpha i_\alpha}{X}_{j_\alpha})\otimes(\overset{\alpha j_\alpha}{g}_{i_\alpha}{}''\otimes\mathrm{id})(R^{(+)})\otimes(\overset{\alpha j_\alpha}{g}_{i_\alpha}{}'\otimes\mathrm{id})(R^{(-)})$$

acts in $\overset{\beta}{V}\otimes\overset{\gamma}{V}\otimes\overset{\delta}{V}\otimes\overset{\varepsilon}{V}$. Recall the notation of 7.2.2. Let r be the r-form of $\mathrm{Pol}(SU_q(2))$ and $\hat{r} = \rho_{21}^{-1}$. We have: (we omit the $\otimes$ symbols, unless we really need them, to simplify the notation)

$$\left\langle \overset{\beta}{v}{}^{i_\beta}\overset{\gamma}{v}{}^{i_\gamma}\overset{\delta}{v}{}^{i_\delta}\overset{\varepsilon}{v}{}^{i_\varepsilon} \left| \sum_{\alpha\in\mathbb{N}} \sum_{\binom{\alpha j_\alpha}{g_{i_\alpha}}} \Delta(\overset{\alpha}{X}{}^{i_\alpha}_{j_\alpha}) \otimes (\overset{\alpha}{g}{}^{j_\alpha}_{i_\alpha}{}'' \otimes \mathrm{id})(R^{(+)}) \otimes (\overset{\alpha}{g}{}^{j_\alpha}_{i_\alpha}{}' \otimes \mathrm{id})(R^{(-)}) \right| \overset{\beta}{v}_{j_\beta}\overset{\gamma}{v}_{j_\gamma}\overset{\delta}{v}_{j_\delta}\overset{\varepsilon}{v}_{j_\varepsilon} \right\rangle$$

$$= \left\langle \overset{\beta}{v}{}^{i_\beta}\overset{\gamma}{v}{}^{i_\gamma} \left| \sum_{\alpha\in\mathbb{N}} \sum_{\binom{\alpha j_\alpha}{g_{i_\alpha}}} \Delta(\overset{\alpha}{X}{}^{i_\alpha}_{j_\alpha}) \left\langle (\overset{\alpha}{g}{}^{j_\alpha}_{i_\alpha}{}'' \otimes \mathrm{id})(R^{(+)}), \overset{\delta}{g}{}^{i_\delta}_{j_\delta} \right\rangle \left\langle (\overset{\alpha}{g}{}^{j_\alpha}_{i_\alpha}{}' \otimes \mathrm{id})(R^{(-)}), \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon} \right\rangle \right| \overset{\beta}{v}_{j_\beta}\overset{\gamma}{v}_{j_\gamma} \right\rangle$$

$$= \left\langle \overset{\beta}{v}{}^{i_\beta}\overset{\gamma}{v}{}^{i_\gamma} \left| \sum_{\alpha\in\mathbb{N}} \sum_{\binom{\alpha j_\alpha}{g_{i_\alpha}}} \Delta(\overset{\alpha}{X}{}^{i_\alpha}_{j_\alpha}) r(\overset{\alpha}{g}{}^{j_\alpha}_{i_\alpha}{}'', \overset{\delta}{g}{}^{i_\delta}_{j_\delta}) r_{21}^{-1}(\overset{\alpha}{g}{}^{j_\alpha}_{i_\alpha}{}', \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon}) \right| \overset{\beta}{v}_{j_\beta}\overset{\gamma}{v}_{j_\gamma} \right\rangle$$

$$= \left\langle \overset{\beta}{v}{}^{i_\beta}\overset{\gamma}{v}{}^{i_\gamma} \left| \sum_{\alpha\in\mathbb{N}} \Delta(\overset{\alpha}{X}{}^{i_\alpha}_{j_\alpha})\ \hat{r}_{13} r_{12}(\overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha}, \overset{\delta}{g}{}^{i_\delta}_{j_\delta}, \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon}) \right| \overset{\beta}{v}_{i_\beta}\overset{\gamma}{v}_{i_\gamma} \right\rangle$$

$$= \sum_{\alpha\in\mathbb{N}} \left\langle \overset{\beta}{v}{}^{i_\beta}\overset{\gamma}{v}{}^{i_\gamma} | \Delta(\overset{\alpha}{X}{}^{i_\alpha}_{j_\alpha}) | \overset{\beta}{v}_{i_\beta}\overset{\gamma}{v}_{i_\gamma} \right\rangle \hat{r}_{13} r_{12}(\overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha}, \overset{\delta}{g}{}^{i_\delta}_{j_\delta}, \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon})$$

Now notice that

$$\sum_{\alpha\in\mathbb{N}} \left\langle \overset{\beta}{v}{}^{i_\beta}\overset{\gamma}{v}{}^{i_\gamma} | \Delta(\overset{\alpha}{X}{}^{i_\alpha}_{j_\alpha}) | \overset{\beta}{v}_{i_\beta}\overset{\gamma}{v}_{i_\gamma} \right\rangle \overset{\alpha}{g}{}^{i_\alpha}_{j_\alpha} = \overset{\beta}{g}{}^{i_\beta}_{j_\beta}\overset{\gamma}{g}{}^{i_\gamma}_{j_\gamma}$$

Thus the last term simplifies as

$$
\begin{aligned}
&\hat{r}_{13} r_{12}(\overset{\beta}{g}{}^{i_\beta}_{j_\beta}\overset{\gamma}{g}{}^{i_\gamma}_{j_\gamma}, \overset{\delta}{g}{}^{i_\delta}_{j_\delta}, \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon}) \\
&= \sum_{\binom{\beta i_\beta}{g_{j_\beta}},\binom{\gamma i_\gamma}{g_{j_\gamma}}} \hat{r}(\overset{\beta}{g}{}^{i_\beta}_{j_\beta}{}'\overset{\gamma}{g}{}^{i_\gamma}_{j_\gamma}{}', \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon}) r(\overset{\beta}{g}{}^{i_\beta}_{j_\beta}{}''\overset{\gamma}{g}{}^{i_\gamma}_{j_\gamma}{}'', \overset{\delta}{g}{}^{i_\delta}_{j_\delta}) \\
&= \sum_{\binom{\beta i_\beta}{g_{j_\beta}},\binom{\gamma i_\gamma}{g_{j_\gamma}},\binom{\delta i_\delta}{g_{j_\delta}},\binom{\varepsilon i_\varepsilon}{g_{j_\varepsilon}}} \hat{r}(\overset{\beta}{g}{}^{i_\beta}_{j_\beta}{}', \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon}{}')\hat{r}(\overset{\gamma}{g}{}^{i_\gamma}_{j_\gamma}{}', \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon}{}'') r(\overset{\beta}{g}{}^{i_\beta}_{j_\beta}{}'', \overset{\delta}{g}{}^{i_\delta}_{j_\delta}{}') r(\overset{\gamma}{g}{}^{i_\gamma}_{j_\gamma}{}'', \overset{\delta}{g}{}^{i_\delta}_{j_\delta}{}'') \\
&= \hat{r}_{14}\hat{r}_{24} r_{13} r_{23}(\overset{\beta}{g}{}^{i_\beta}_{j_\beta}, \overset{\gamma}{g}{}^{i_\gamma}_{j_\gamma}, \overset{\delta}{g}{}^{i_\delta}_{j_\delta}, \overset{\varepsilon}{g}{}^{i_\varepsilon}_{j_\varepsilon}) \\
&= \left\langle \overset{\beta}{v}{}^{i_\beta}\overset{\gamma}{v}{}^{i_\gamma}\overset{\delta}{v}{}^{i_\delta}\overset{\varepsilon}{v}{}^{i_\varepsilon} \left| R^{(-)}_{14} R^{(-)}_{24} R^{(+)}_{13} R^{(+)}_{23} \right| \overset{\beta}{v}_{i_\beta}\overset{\gamma}{v}_{i_\gamma}\overset{\delta}{v}_{i_\delta}\overset{\varepsilon}{v}_{i_\varepsilon} \right\rangle
\end{aligned}
$$

This finishes the proof of the lemma.

The group like element of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ was defined as: $G = q^{2J_z}$, in particular $\phi(G) = q^{2J_z} \otimes q^{2Jz} = \hat{G}$, where $\hat{G}$ is the group like element of $U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$. Thus in the conditions of the last theorem we also have: $f(Gx) = \hat{G}f(x)$. By the discussion in 7.2.4, the map $\phi : U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}}) \to U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$ also preserves the comultiplication map and the antipodal map. Since the category of representations of $U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$ which are direct sums of representations $\overset{\alpha}{\rho} \otimes \overset{\beta}{\rho}$ is a ribbon category, it follows that:

Theorem 118 . *The tensor category of finite dimensional representations of the quantum Lorentz Group generated by the balanced representations is a ribbon category when the Quantum Lorentz Group is given the formal R-matrix $\mathcal{R}$ and group like element $G = q^{2J}$. It is ribbon equivalent to the subcategory of representations of $U_q(\mathfrak{su}(2))\hat{\otimes}U_q(\mathfrak{su}(2))$ generated by representations $\overset{\alpha}{\rho} \otimes \overset{\alpha}{\rho}$*

Therefore we have a knot invariant $I(\rho(p)_{\text{fin}})$ for any finite dimensional balanced representation $\rho(p)_{\text{fin}}$ of the quantum Lorentz group made out of the formal R-matrix and group like elements. Moreover by, Lemma 101, it follows that:

Theorem 119 *Let $q \in (0,1)$. Let $p \in \mathbb{N}$ and $2\alpha = p-1$. We have:*

$$I(\rho(p)_{\text{fin}})(q^{1/2})(K) = X(0,p,K)(-h)(2\alpha+1),$$

where $q = \exp(h)$ and $h < 0$, for any knot K. In particular these knot invariants are unframed and do not distinguish a knot from its mirror image.

This proves our perturbative framework in chapter 2 is correct at least for finite dimensional balanced representations. We will see later that the same holds for infinite dimensional balanced representations.

7.4 Knot invariants from infinite dimensional representations of the Quantum Lorentz Group

We now apply the machinery of the last sections to define knot invariants from infinite dimensional representations of the Quantum Lorentz Group.

7.4.1 Representations of the Quantum Lorentz Group and R-matrix-a resume of the notation

Recall that the Quantum Lorentz Group $\mathcal{D} = U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ at a point $q \in (0,1)$ is defined as the quantum double $\mathcal{D}(U_q(\mathfrak{su}(2)), \text{Pol}(SU_q(2))^{\text{cop}})$, notice that both $U_q(\mathfrak{su}(2))$ and $\text{Pol}(SU_q(2))^{\text{cop}}$ are sub Hopf algebras of $\mathcal{D}$. Therefore $\mathcal{D}$ has a *formal* R-matrix $\mathcal{R}$ coming from its quantum double structure. As we have seen in 7.3.7, even though $\mathcal{R}$ is defined by an infinite sum, it is possible to describe its action in any pair of infinite dimensional irreducible balanced representations of $\mathcal{D}$, as well as in the balanced finite dimensional representations. See 7.3.7

Recall that the balanced representations of $U_q(\mathfrak{sl}(2,\mathbb{C})_{\mathbb{R}})$ depend on $p \in \mathbb{C}$. If $p \in i\mathbb{R}$ then the representations $\rho(p)$ can be made unitary. Choosing $p \in [0, i\frac{2\pi}{h}]$, where $q = e^h$, parametrises all the unitary representations in the principal series which have minimal spin equal to zero. These last ones are called simple representations in [NR].

Similarly to the classical case, the underlying vector space for the balanced representations $\rho(p), p \in \mathbb{C}$ of the Quantum Lorentz Group is

$$V = V(p) = \bigoplus_{\alpha \in \mathbb{N}_0} \overset{\alpha}{V},$$

where

$$\overset{\alpha}{\rho} : U_q(\mathfrak{su}(2)) \to L(\overset{\alpha}{V})$$

is the irreducible representation of $U_q(\mathfrak{su}(2))$ with spin α. A basis of $\overset{\alpha}{V}$ is thus given by the vectors $\{\overset{\alpha}{u}_i, i = -\alpha, -\alpha+1, ..., \alpha\}$. Any element x of $U_q(\mathfrak{su}(2))$ acts in V in the fashion:

$$\prod_{\alpha \in \mathbb{N}_0} \overset{\alpha}{\rho}(x).$$

The group like element of the Lorentz group is given by $G = q^{2J_z}$.

Define, given half integers A, B, C and D, the complex numbers

$$\Lambda^{BC}_{AD}(\rho) = \sum_{\sigma} \left(\begin{array}{c|cc} 0 & C & B \\ A & \sigma & -\sigma \end{array} \right) q^{2\sigma\rho} \left(\begin{array}{cc|c} -\sigma & \sigma & D \\ B & C & 0 \end{array} \right). \tag{168}$$

For the correct definition of the phases of the Clebsh-Gordon coefficients see [BR2] or [KS]. We displayed an expression for them in 7.1.2, a formula taken from [BR2].

Recall the formal universal R-matrix of the quantum Lorentz group is: (see [BR1] and 7.3.7)

$$\mathcal{R} = \sum_{\substack{\alpha\in\frac{1}{2}\mathbb{N}_0\\ -\alpha\le i_a, j_a\le\alpha}} \overset{\alpha i_\alpha}{X}_{j_\alpha} \otimes \overset{\alpha j_\alpha}{g}_{i_\alpha},$$

its inverse being:

$$\mathcal{R}^{-1} = \sum_{\substack{\alpha\in\frac{1}{2}\mathbb{N}_0\\ -\alpha\le i_a, j_a\le\alpha}} \overset{\alpha i_\alpha}{X}_{j_\alpha} \otimes S^{-1}(\overset{\alpha j_\alpha}{g}_{i_\alpha}).$$

This antipode S is the one of $\mathrm{Pol}(SU_q(2))^{\mathrm{cop}} \subset \mathcal{D}$, which is the inverse of the one in $\mathrm{Pol}(SU_q(2))$, thus

$$S(\overset{\alpha j_\alpha}{g}_{i_\alpha}) = q^{-i_\alpha+j_\alpha}(-1)^{-j_\alpha+i_\alpha}\overset{\alpha -i_\alpha}{g}_{-j_\alpha}.$$

See 7.1.4 The action of $\overset{\alpha i_\alpha}{g}_{j_\alpha}$ in the space $V(p)$ is given by

$$\overset{\alpha i_\alpha}{g}_{j_\alpha}\overset{\beta}{v}_{i_\beta} = \frac{\mathcal{F}_\beta}{\mathcal{F}_\gamma}\sum_{D,\gamma,p,r,x\frac{1}{2}\mathbb{N}_0} \overset{\gamma}{v}_{i_\gamma} \left(\begin{array}{cc|c} i_\gamma & i_\alpha & D \\ \gamma & \alpha & x \end{array}\right)\left(\begin{array}{c|cc} x & \alpha & \beta \\ D & j_\alpha & i_\beta \end{array}\right)\Lambda^{\alpha C}_{\gamma\beta}. \tag{169}$$

The constants $\mathcal{F}_\alpha$ are defined in [BR2], proposition 1. They will not be used directly. In fact as we refered to in 7.3.3, their values are (almost) arbitrary and they only appear to ensure that the representations $\rho(p)$ are unitary for $p \in i\mathbb{R}$. Their appearance does not change the representation itself, therefore not affecting the calculations of knot invariants. Notice also that equation (169) implies that $\overset{\frac{1}{2} i}{g}_j$ sends $\overset{\gamma}{V}$ to $\overset{\gamma-1}{V} \oplus \overset{\gamma}{V} \oplus \overset{\gamma+1}{V}$.

In some particular cases, equation (169) simplifies to:

$$\overset{\alpha i_\alpha}{g}_{j_\alpha}\overset{0}{v}_0 = \frac{\mathcal{F}_0}{\mathcal{F}_\gamma}\sum_{\gamma, i_\gamma} \left(\begin{array}{cc|c} i_\gamma & i_\alpha & \alpha \\ \gamma & \alpha & j_\alpha \end{array}\right)\Lambda^{\alpha\alpha}_{\gamma 0}\overset{\gamma}{v}_{i_\gamma}, \tag{170}$$

and to

$$\left\langle \overset{0}{v}^{0}, \overset{\alpha i_\alpha}{g}_{j_\alpha}\overset{\beta}{v}_{i_\beta} \right\rangle = \frac{\mathcal{F}_b}{\mathcal{F}_0}\left(\begin{array}{c|cc} i_\alpha & \alpha & \beta \\ \alpha & j_\alpha & i_\beta \end{array}\right)\Lambda^{\alpha\alpha}_{0\beta}. \tag{171}$$

All these formulae are consequences of (108) and (109). With them we can also prove $\Lambda^{0\alpha}_{\alpha\alpha} = 1$, from which follows:

$$\overset{00}{g_0}\overset{\alpha}{v}_{i_\alpha} = \overset{\alpha}{v}_{i_\alpha}. \tag{172}$$

As we saw in 7.3.7, the elements

$$\overset{\alpha i_\alpha}{X_{j_\alpha}}, \alpha \in \frac{1}{2}\mathbb{N}_0, i_\alpha = -\alpha, ..., \alpha$$

act simply as matrix elements, in other words:

$$\overset{\alpha i_\alpha}{X_{j_\alpha}}\overset{\beta}{v}_{i_\beta} = \delta(\alpha,\beta)\delta(i_a, i_\beta)\overset{\alpha}{v}_{j_\alpha}. \tag{173}$$

The action of the group like element G is

$$G\overset{\alpha}{v}_{i_\alpha} = q^{2i_a}\overset{\alpha}{v}_{i_\alpha}. \tag{174}$$

It is easy to compute how $\mathcal{R}$ acts, namely:

$$\mathcal{R}\left(\overset{\alpha}{v}_{i_\alpha} \otimes \overset{\beta}{v}_{i_\beta}\right) = \sum_{D,\gamma,j_\alpha} \left(\begin{matrix} i_\gamma & j_\alpha \\ \gamma & \alpha \end{matrix} \middle| \begin{matrix} D \\ x \end{matrix}\right) \left(\begin{matrix} x \\ D \end{matrix} \middle| \begin{matrix} \alpha & \beta \\ i_\alpha & i_\beta \end{matrix}\right) \frac{\mathcal{F}_\beta}{\mathcal{F}_\gamma}\Lambda^{\alpha D}_{\gamma\beta}\left(\overset{\alpha}{v}_{j_\alpha} \otimes \overset{\gamma}{v}_{i_\gamma}\right).$$

See [BR1], proposition 13. The action of $\mathcal{R}$ in $V \otimes V$ is thus well defined. Notice that we are considering the algebraic, rather than topological, tensor product. Moreover $\mathcal{R}$ defines a braid group representation. We proved this in 7.3.7. Denote it by $b \in B(n) \mapsto R_b \in L(V^{\otimes})$. Here $B(n)$ denotes the n-strand braid group and $L(V^{\otimes n})$ the vector space of linear maps $V^{\otimes n} \to V^{\otimes n}$. Notice that the braiding operators R_b extend to unitary operators if $p \in i\mathbb{R}$ since $\mathcal{R}^{*\otimes *} = \mathcal{R}^{-1}$, where $*$ is the star structure on the Quantum Lorentz Group, see 7.3.7.

7.4.2 Some heuristics

Recall 1.4.2 as well now. Suppose we are given a braid b with $n+1$ strands, there is attached to it q braiding operator $R_b : V^{\otimes(n+1)} \to V^{\otimes(n+1)}$, for any balanced representation of the Quantum Lorentz Group. Consider the map $A_b = (\mathrm{id} \otimes G \otimes ... \otimes G)R_b$. Suppose the closure of the braid b is a knot. If the representations we are considering were finite dimensional then the partial trace $T^1(A_b) : V \to V$ of A_b over the last n variables would be an intertwiner, and thus a multiple of the identity, given that the representations which we are considering are irreducible. Moreover *this* multiple of the identity would be a knot invariant which would have the form:

$$b \mapsto \sum_{\alpha_1,...,a_n \in \mathbb{N}_0} \left\langle \overset{0}{v}^0 \otimes \overset{\alpha_1}{v}^{i_{\alpha_1}} \otimes ... \otimes \overset{\alpha_n}{v}^{i_{\alpha_n}}, A_b \left(\overset{0}{v}_0 \otimes \overset{\alpha_1}{v}_{i_{\alpha_1}} \otimes ... \otimes \overset{\alpha_n}{v}_{i_{\alpha_n}} \right) \right\rangle. \tag{175}$$

The last sum is to be also extended to all i_{α_k} with $-\alpha_k \leq i_{\alpha_k} \leq \alpha_k, k = 1, ..., n-1$. Even though the sums above may be not convergent, the assignment of one sum of this kind to a braid whose closure is a knot is not ambiguous. In fact suppose b has $m+1$ strands and n crossings. We can always express this sum in a more sugestive way, namely as:

$$S_b = \sum_{\substack{a_1,...,\alpha_n \\ -\alpha_l \leq i_l, j_l \leq \alpha_l}} \left\langle \overset{0}{v}^0, \prod_{l=1}^{2n+m} T(\underline{\alpha}, \underline{i}, \underline{j}, l) \overset{0}{v}_0 \right\rangle, \tag{176}$$

where $\underline{\alpha} = (\alpha_1, ..., \alpha_n)$, $\underline{i} = (i_1, ..., i_n)$, $\underline{j} = (j_1, ..., j_n)$, and $T(\underline{\alpha}, \underline{i}, \underline{j}, l)$ can be either a term like $\overset{\alpha_k}{g}{}^{i}{}_{j}$, a term like $\overset{\alpha_k i}{X}{}_{j}$ or G. We obvioulsy need to suppose that the closure of b is a knot for this to hold. Notice that the transition from (175) to (176) is totaly clear if the representations are finite dimensional, and corresponds to the approach in 1.4.2. We wish to analyse the sums $b \mapsto S_b$ where b is a braid whose closure is a knot. We shall see that they do define a knot invariant after we consider their asymptotic expansion.

Figure 32: Right and Left Handed Trefoil Knot

Let us look to the sums S_b in a bit more detail for two explicit example, clarifying what the formula for S_b means as well. We consider first the left and right handed trefoil knots displayed in figure 32. Call the two braids we have chosen to represent them T_+ and T_-. The sum for the right handed trefoil knot is

$$S_{T_+} = \sum_{\alpha,\beta,\gamma\in\frac{1}{2}\mathbb{N}_0} \left\langle \overset{0^0}{v}, \overset{\gamma i_\gamma}{g}_{j_\gamma} \overset{\beta i_\beta}{X}_{j_\beta} \overset{\alpha j_\alpha}{g}_{i_\alpha} G \overset{\gamma i_\gamma}{X}_{j_\gamma} \overset{\alpha j_\beta}{g}_{i_\beta} \overset{\alpha i_\alpha}{X}_{j_\alpha} \overset{0}{v}_0 \right\rangle .$$

Whereas for the left handed trefoil is:

$$S_{T_-} = \sum_{\alpha,\beta,\gamma\in\frac{1}{2}\mathbb{N}_0} \left\langle \overset{0^0}{v}, \overset{\gamma i_\gamma}{X}_{j_\gamma} S(\overset{\alpha j_\beta}{g}_{i_\beta}) \overset{\alpha i_\alpha}{X}_{j_\alpha} G S(\overset{\alpha j_\gamma}{g}_{i_\gamma}) \overset{\beta i_\beta}{X}_{j_\beta} S(\overset{\alpha j_\alpha}{g}_{i_\alpha}) \overset{0}{v}_0 \right\rangle .$$

Many of the terms will be zero in the expressions above. Let us look at S_{T_-}. We only want the $0 \to 0$ matrix element, and $\left\langle \overset{0^0}{v}, \overset{\alpha i_\alpha}{X}_{j_\alpha} v \right\rangle = \delta(\alpha, 0) \left\langle \overset{0^0}{v}, v \right\rangle$. Thus we can make $\gamma = 0$, and then note that $\overset{0^0}{g}_0$ acts as the identity. We obtain

$$S_{T_-} = \sum_{\alpha,\beta\in\frac{1}{2}\mathbb{N}_0} q^{-i_\alpha+j_\alpha-i_\beta+j_\beta}(-1)^{i_\alpha+i_\beta-j_\alpha-j_\beta} < \overset{\beta-i_\beta}{g}_{-j_\beta} \overset{\alpha i_\alpha}{X}_{j_\alpha} G \overset{\beta i_\beta}{X}_{j_\beta} \overset{\alpha-i_\alpha}{g}_{-j_a} \overset{0}{v}_0, \overset{0}{v}_0 > .$$

From (162) and (174) follows $\alpha = \beta$ and $i_\alpha = j_\beta$. By (170) and (171) we can conclude:

$$S_{T_-} = \sum_{\alpha \in \frac{1}{2}\mathbb{N}_0} \sum_{i_\beta, j_\beta, j_\alpha = -\alpha}^{\alpha} q^{j_\alpha - i_\beta + 2j_\beta} (-1)^{i_\beta - j_\beta}$$
$$\left(\begin{array}{c|cc} -i_\beta & \alpha & \alpha \\ \alpha & -j_\beta & j_\alpha \end{array}\right) \left(\begin{array}{cc|c} i_\beta & -j_\beta & \alpha \\ \alpha & \alpha & -j_\alpha \end{array}\right) \Lambda^{\alpha\alpha}_{0\alpha} \Lambda^{\alpha\alpha}_{\alpha 0}.$$

Using the standard symmetries of the Clebsch Gordon Coefficients, namely equation (114), we can express this as:

$$S_{T_-} = \sum_{\alpha \in \mathbb{N}_0} \sum_{i_\beta, j_\beta, j_\alpha = -\alpha}^{\alpha} q^{2j_\beta} \left(\begin{array}{cc|c} -i_\beta & -j_\alpha & \alpha \\ \alpha & \alpha & -j_\beta \end{array}\right) \left(\begin{array}{c|cc} -j_\beta & \alpha & \alpha \\ \alpha & -i_\beta & -j_\alpha \end{array}\right) \Lambda^{\alpha\alpha}_{0\alpha} \Lambda^{\alpha\alpha}_{\alpha 0}.$$

Notice $\Lambda^{\alpha\alpha}_{\alpha 0}$ is zero unless α is integer. Therefore the final expression for the sum is:

$$S_{T_-} = \sum_{\alpha \in \mathbb{N}_0} d_\alpha \Lambda^{\alpha\alpha}_{0\alpha} \Lambda^{\alpha\alpha}_{\alpha 0}.$$

Here d_a is the quantum dimension of the representation $\overset{\alpha}{\rho}$. It equals $(q^{2\alpha+1} - q^{-2\alpha-1})/(q - q^{-1})$. This last sum is easily proved to be equal to S_{T_+}, therefore, if the sums do define a knot invariant, they make no distinction between the Trefoil and its mirror image. We would expect this from Corollary 19 and Theorem 119. The calculations for other knot diagrams follow the same procedure, which can be given an obvious graphical calculus.

Notice that the series $S(T_-)$ seems to be divergent due to the presence of the d_α term in it. Therefore these sums do not seem to define $\mathbb{C}$-valued knot invariants. We will go back to this in 7.4.5. This tells us the method of Borel re-summation sketched in Chapter 4 is perhaps more powerful.

As we have seen in 7.3.5, if $p \in \mathbb{N}$, then the representation $\rho(p)$ has a finite dimensional subrepresentation $\rho(p)_{\text{fin}}$ in $V(p)_{\text{fin}} = \overset{0}{V} \otimes \overset{1}{V} \oplus \ldots \oplus \overset{p-1}{V}$. Notice that $\overset{0}{v}_0 \in V(p)_{\text{fin}}$. Therefore from (173) and (176) we have:

Lemma 120 *If $p \in \mathbb{N}$ and b is a braid whose closure is a knot, then the infinite sum $S_b(q,p)$ truncates to a finite sum for any $q \in (0,1)$.*

And by Theorem 119 we must have:

Proposition 121 *Let $q = exp(h/2) \in (0,1)$, and $p \in \mathbb{N}$. Let also $\alpha = (p-1)/2$. Given a braid b, let K_b be the closure of b, which we suppose to be a knot. We have:*

$$S_b(\exp(h/2), p)[2\alpha + 1]^2 = X(0, p, K_b)(-h)(2\alpha + 1)^2$$

where $\alpha = (p-1)/2$ and $h < 0$.

Recall that by theorem 14 the knot invariant $X(0,p)$ is unframed for any $p \in \mathbb{N}$.

7.4.3 The series are convergent h-adicaly

We now define a simple version the h-adic version of the theory developed by Buffenoir and Roche[18]

Let $\sum_n a_n h^n$ be a formal power series. The order of it is by definition the smallest n so that $a_n \neq 0$. Let $\sum_k A(k) = \sum_k \left(\sum_n a_n^{(k)} h^n\right)$ be a series of formal power series.

[18]Maybe it would have been better to define an h-adic quantum Lorentz group as $\mathcal{D}(U_h(\mathfrak{sl}(2,\mathbb{C})), SL_h(2,\mathbb{C}))$ from the begining. This quantum Group which would then naturally map to $U_h(\mathfrak{sl}(2,\mathbb{C})) \otimes U_h(\mathfrak{sl}(2,\mathbb{C}))$. This would avoid the technicalities arising from the fact $U_q(\mathfrak{su}(2))$ is not an honest ribbon Hopf algebra. However, we could not, a priory, apply Takeuchi's theorem (Theorem 103), a result in which a major part of our work depends.

We say it is converges h-adically if for any $m \in \mathbb{N}$ the number of k's in $\mathbb{N}$ such that the power series $A(k)$ has order smaller than m is finite. Therefore we can define $\sum_k \left(\sum_n a_n^{(k)} h^n\right)$ without any problem.

Let $q \in (0,1)$ and consider the element $\overset{\alpha i_\alpha}{g}_{j_\alpha} \in \mathrm{Pol}(SU_q(2))$. For any $p \in \mathbb{C}$, we have a balanced representations $\rho(p)$ of the quantum Lorentz group in $V(p)$. The term

$$\left\langle \overset{\beta i_\beta}{v} | \rho(p)(\overset{\alpha i_\alpha}{g}_{j_\alpha}) | \overset{\gamma}{v}_{i_\gamma} \right\rangle_q$$

can be seen a function of q. Due to the fact the building blocks of $\rho(p)$ are Clebsch Gordon coefficients, it express as a sum of square roots of rational functions of q, which extend to a well defined analytic function in a neighborhood of 1. We can see this for example from (107). In addition we have some terms of the form $q^{p\sigma}, \sigma \in \mathbb{Z}$, which after putting $q = \exp(h/2)$ define an analytic function of h. Therefore

$$h \mapsto \left\langle \overset{\beta i_\beta}{v} | \rho(p)(\overset{\alpha i_\alpha}{g}_{j_\alpha}) | \overset{\gamma}{v}_{i_\gamma} \right\rangle_{\exp(h/2)}$$

defines a power series in h, uniquely. In particular it follows that if b is a braid whose closure is a knot, then each term of the sum S_b defines uniquely a power series in h, which converges to the term for $h \leq 0$ close enough to 0 and q close enough to 1.

Lemma 122 *For any $x \in \mathrm{Pol}(SU_q(2))$, the order of*

$$h \mapsto \left\langle \overset{\beta i_\beta}{v} | \rho(p)(x) | \overset{\gamma}{v}_{i_\gamma} \right\rangle_{\exp(h/2)}$$

is bigger or equal to $|\beta - \gamma|$ as a power series in h.

Proof. Notice that $\overset{\frac{1}{2} i}{g}_j$ sends $\overset{\gamma}{V}$ to $\overset{\gamma-1}{V} \oplus \overset{\gamma}{V} \oplus \overset{\gamma+1}{V}$, in a way such that for $q = 1$ the projection v of $\overset{\frac{1}{2} i}{g}_j \overset{\gamma}{v}_{i_\gamma}$ in $\overset{\gamma+1}{V} \oplus \overset{\gamma-1}{V}$ is zero. We can see this from Lemma 109, for example. In particular v has order bigger or equal to one. This lemma is thus a trivial consequence of the fact the elements $\left\{ \overset{\frac{1}{2}i}{g}_j, -1/2 \leq i, j \leq 1/2 \right\}$ generate $\mathrm{Pol}(SU_q(2))$ as an algebra. ■

Therefore

Proposition 123 *For any braid b whose closure is a knot the sum S_b converges to an element of $\mathbb{C}[[h]]$ in the h-adic topology.*

Proof. Let b be a braid with n crossings and $m+1$ strands, thus

$$S_b = \sum_{\substack{a_1,\dots,\alpha_n \\ -\alpha_l \leq i_l, j_l \leq \alpha_l}} \left\langle \overset{0}{v}^0, \prod_{l=1}^{2n+m} T(\underline{\alpha}, \underline{i}, \underline{j}, l) \overset{0}{v}_0 \right\rangle,$$

where $\underline{\alpha} = (\alpha_1, ..., \alpha_n)$, $\underline{i} = (i_1, ..., i_n)$, $\underline{j} = (j_1, ..., j_n)$, and $T(\underline{\alpha}, \underline{i}, \underline{j}, l)$ can be either a term like $\overset{\alpha_k i}{g}_j$, a term like $\overset{\alpha_k i}{X}_j$ or G. Recall that we need to suppose that the closure of b is a knot for this to hold. Due to the way the $\overset{\alpha_k i}{X}_j$ act in $V(p)$, see equation (173), the previous lemma guaranties that the order of $\left\langle \overset{0}{v}^0, \prod_{l=1}^{2n+m} T(\underline{\alpha}, \underline{i}, \underline{j}, l) \overset{0}{v}_0 \right\rangle$ as a power series is bigger than or equal to α_k for $k = 1, .., n$, and the result follows. ■

7.4.4 The series define a $\mathbb{C}[[h]]$-valued knot invariant

Since we have proved the h-adic convergence of the sums S_b to a formal power series, we could now use Markov's theorem and prove that the assignment $b \mapsto S_b$ defines a $\mathbb{C}[[h]]$-valued knot invariant. However the best way to prove it is to reduce it to the finite dimensional case, since we already know that it defines a knot invariant and what the form of it is, see Proposition 121. Consider a coefficient $\Lambda_{AD}^{BC}(p)_q$ at $q = \exp(h/2)$, thus it is a power series in h convergent for h small enough. From equation (154), we can see that the dependence of each term of it in p is polynomial. In particular:

Lemma 124 *Let b be a braid whose closure is a knot K_b, consider the power series $S(b)(\exp(h/2), p)$ as a function of p, the parameter defining a balanced representation of the quantum Lorentz Group. Then each term in the expansion of $S(b)$ is a polynomial in p.*

Proof. Cf 3.3.1. Suppose $A(p) = \sum_{n\in\mathbb{N}_0} A_n(p)h^n$ and $B(p) = \sum_{n\in\mathbb{N}_0} B_n(p)h^n$ are power series whose coefficients depend polynomially in p, for example a power series such as $\exp(mph/2)$. Then also the coefficients of their product depend polynomially in p. This immediately proves this lemma since the Clebsch-Gordan coefficients do not depend on p. ∎

Therefore

Theorem 125 *Let $p \in \mathbb{C}$ and b be a braid whose closure is a knot K_b. We have:*

$$S_b(\exp(h/2), p)[2a+1]^2 = X(0, p, K_b)(-h)(2\alpha+1)^2$$

where $\alpha = (p-1)/2$

Recall that by theorem 14 the knot invariant $X(0, p)$ is unframed.

Proof. By Lemma 124, we only need to prove this theorem for $p \in \mathbb{N}$. In this case, if $q = (0, 1)$ then $S(p, q)$ truncates to a finite sum which from Proposition 121 equals $X(0, p, K_b)(-h)\frac{(2\alpha+1)^2}{[2\alpha+1]^2}$ at $h = \exp(h/2)$. Recall these power series are convergent if p is integer. Each term of the finite sum $S(p, \exp(h/2))$ is a power series in h convergent for h small enough and coinciding with $S(p, q)$, for $q \in (0, 1)$ and close enough to 1; thus the result follows. ∎

This theorem totally unifies the approach in chapter 2 and this one in the context of balanced representations.

7.4.5 Estimates for Clebsch Gordon Coefficients and Λ-coefficients, and the series S_{T_+}

We now estimate how fast the series S_{T_+} can diverge, now interpreted as a series of complex numbers.

First of all we make the observation that the orthogonality relations for the Clebsch Gordon coefficients imply that they are in norm smaller than one, for they are real in our convention, which is as usual the one of [BR2], see 7.1.2. We can actually get a better estimate. First, let us try to find a bound for a Clebsch-Gordan coefficient of the form $\left(\begin{array}{cc|c} \sigma & -\sigma & D \\ B & C & 0 \end{array}\right)$. Notice that in general.

$$\left(\begin{array}{c|cc} 0 & C & B \\ A & -\sigma & \sigma \end{array}\right) = \left(\begin{array}{cc|c} -\sigma & \sigma & A \\ C & B & 0 \end{array}\right) = \left(\begin{array}{cc|c} -\sigma & \sigma & A \\ B & C & 0 \end{array}\right). \tag{177}$$

See 7.1.2.

We have, see [BR2] and 7.1.2:

$$\left(\begin{array}{cc|c} \sigma & -\sigma & D \\ B & C & 0 \end{array}\right) = q^{\sigma+\frac{1}{2}((C+1)C-(B+1)B-(D+1)D)}.e^{i\pi(B-\sigma)}$$
$$\sqrt{\frac{[2D+1][B+C-D]![B-\sigma]![D]![D]!}{[D+C-B]!B+D-C]![B+C+D+1]![B+\sigma]![C-\sigma!}}$$
$$\sum_{\substack{V=0 \\ D-C-\sigma\leq V\leq B-\sigma}}^{D} \frac{Q^{V(D+1)}e^{i\pi(V+B-\sigma)}[B+\sigma+V]![C+D-\sigma-V]!}{[V]![D-V]![B-\sigma-V]![C-D+\sigma+V]!}.$$

The quantum integers are defined as $[n]=[n]_q=\frac{q^{-n}-q^n}{q^{-1}-q}$. As usual we assume $q\in(0,1)$ Note that we can make the substitution $[n]_q\mapsto q^{-n}-q^n$ in the formula above for Clebsch-Gordan Coefficients, at the price of multiplying the final expression by $(q^{-1}-q)^{-\frac{1}{2}}$, as an easy calculation shows. We then propose to consider the transition $q^{-n}-q^{n}\mapsto q^{-n}$. Let us analyse first each term of the final sum. A straightforward calculation shows that we can make the transition if we multiply each term by K_V, where

$$K_V=\frac{F(B+\sigma+V)F(C+D-\sigma-V)}{F(V)F(D-V)F(B-\sigma-V)F(C-D+\sigma+V)}.$$

We considered $F(0)=1, F(n)=\prod_{k=1}^{n}(q^{2k}-1), n=1,2,....$ Since $\prod_{k=1}^{\infty}(q^{2k}-1)$ is convergent we conclude that K_V is limited by a constant for any choice of parameters. If we do the same with the other terms we can prove:

Theorem 126 *There exists a constant $K<\infty$ such that for any $B,C\in\frac{1}{2}\mathbb{N}_0, D\in\{|B-C|,...,B+C\}$ and any $\sigma\in\{-\min(B,C),-\min(B,C)+1,...,$
$\min(B,C)\}$ we have:*

$$\left|\begin{pmatrix}\sigma & -\sigma & \Big| & D\\ B & C & \Big| & 0\end{pmatrix}\right|\leq Kq^{C^2+C}.q^{-BC}.q^{D(B-C)}.q^{\sigma}q^{\sigma(C-B)}.q^{-D}$$
$$\sum_{\substack{V=0\\ D-C-\sigma\leq V\leq B-\sigma}}^{D} q^{V^2+V(1+2(C-B))}. \quad (178)$$

Using similar techniques we can prove:

Theorem 127 *Let $q\in(0,1)$. There exists a $K<\infty$ such that for any I,J,K,m,n,p we have:*

$$\left|\begin{pmatrix}m & n & \Big| & K\\ I & J & \Big| & p\end{pmatrix}\right|\leq q^{J^2+J-IJ+K(I-J)+mI+nJ+\frac{1}{2}m+\frac{1}{2}n}q^{-K+mp+m}$$
$$\sum_{\substack{V=0\\ K-J-m\leq V\leq I-m}}^{K-p} q^{V^2+V(2p+1-2I+2J)}.$$

Now using the symmetry properties of the Clebsch Gordon Coefficients (177) we conclude

Corollary 128 *We have the bounds*

$$\left|\begin{pmatrix} \sigma & -\sigma & \bigg| & D \\ C & B & \bigg| & 0 \end{pmatrix}\right| \leq Kq^{C^2+C}.q^{-BC}.q^{D(B-C)}.q^{\sigma}q^{\sigma(C-B)}.q^{-D} \sum_{\substack{V=0 \\ D-C-\sigma\leq V\leq B-\sigma}}^{D} q^{V^2+V(1+2(C-B))}, \quad (179)$$

and if $p \in i\mathbb{R}$

$$|\Lambda_{AD}^{BC}(p)| \leq K.q^{2C^2+2C}.q^{-2BC}q^{(A+D)(B-C)}.q^{-(A+D)} \sum_{\sigma=-\min(B,C)}^{\min(B,C)} \left(\sum_{\substack{W=0 \\ A-C+\sigma\leq W\leq B+\sigma}}^{A} q^{W^2+W(1+2C-2B)} + \sum_{\substack{V=0 \\ D-C-\sigma\leq V\leq B-\sigma}}^{D} q^{V^2+V(1+2C-2B)} \right). \quad (180)$$

To prove the second estimate recall that q is real and ρ is imaginary in (154). Notice the estimates in equations (178) and (179) are different. It is not difficult to refine the bound for $\Lambda_{E0}^{\beta\beta}$ which yields.

$$|\Lambda_{E0}^{\beta\beta}| \leq K.(2\beta+1)Eq^{2\beta-E}, E \in \{0,1,...,2\beta\}. \quad (181)$$

Also $\Lambda_{E0}^{\beta\beta} = 0$ if $E > 2\beta$ this is because of properties of $Y(E,\beta,\beta)$ expressing admissibility, see (154). The same is valid for $\Lambda_{0E}^{\beta\beta}$.

Recall

$$S_{T_-} = \sum_{\alpha\in\mathbb{N}_0} \frac{q^{2\alpha+1}-q^{-2\alpha-1}}{q-q^{-1}} \Lambda_{\alpha 0}^{\alpha\alpha}\Lambda_{0\alpha}^{\alpha\alpha}.$$

Therefore if $p \in i\mathbb{R}$ the series diverges at most as[19]:

$$\sum_{\alpha \in \mathbb{N}} (2\alpha + 1).$$

References

[AC] Altschuler D., Coste A., Quasi-Quantum Groups, Knots, Three Manifolds and Topological Field Theory; Commun. Math. Phys., 150,83-107 (1992)

[AF] Altschuler, D., Freidel, L.: On universal Vassiliev Invariants, XIth International Congress of Mathematical Physics (Paris, 1994), 709–710, Internat. Press, Cambridge, MA, 1995.

[BC] Barrett J.W., Crane L.: A Lorentzian Signature Model for Quantum General Relativity, Class. Quant. Grav. 17 (2000) 3101-3118

[BR1] Buffenoir E., Roche Ph: Tensor Product of Principal Unitary Representations of Quantum Lorentz Group and Askey-Wilson Polynomials , Journal of Mathematical Physics 41 (11), 7715-7751, Nov 2000

[BR2] Buffenoir E., Roche Ph., Harmonic Analysis in the Quantum Lorentz Group, Commun. Math. Phys., 207 (3), 499-555 (1999)

[BN] Bar-Natan D.: On the Vassiliev Knot Invariants, Topology Vol. 34 No. 2 (1995) pp 423-472.

[19]Numerical calculation with maple sugest that the series S_{T_+} is indeed divergent and that this is the exact divergence rate

[BNR] Buffenoir,E., Noui, K., Roche, Ph.: Hamiltonian Quantisation of Chern Simons Theory with $SL(2,\mathbb{C})$ group, Class. Quant. Grav. 19 (2002) 4953-5015, hep-th/0202121

[C] Chmutov S.: A Proof of the Melvin Morton Conjecture and Feynmann Diagrams, J. Knot Theory Ramif (7) 1, 23-40, Feb 1998

[CD] Chmutov S., Duzhin S.: The Kontsevich Integral, Acta. Appl. Math 66 (2) 155-190, April 2001

[CH] Chabat B.: Introducion à l'analyse complexe , Tome 1, Editions Mir, 1990

[CP] Chari V, Pressley A: A Guide to Quantum Groups, Cambridge University press 1994

[CV] Chmutov S, Varchenko A., Remarks on The Vassiliev Invariants coming from $\mathfrak{sl}_2$, Topology, Vol 36, No. 1, pp 153-178, 1997

[D1] Drinfeld V. G.: Quasi Hopf Algebras, Leningrad Math. J. **1** (1990) 1419-1457

[D2] Drinfeld V. G: on Quasi Triangular Quasi Hopf Algebras and a Group Closelly related with $Gal(\hat{\mathbb{Q}}/Q)$, Leningrad Math. J. **2** (1991) 829-860

[FL] Freidel L., Louapre D.: Non-perturbative summation over 3D discrete topologies, hep-th/0211026

[FM1] Faria Martins, J: On the Analytic Properties of the z-coloured Jones Polynomial, To appear in Journal of Knot Theory and its ramifications

[FM2] Faria Martins, J:Knot Theory with the Lorentz Group, mathQA/

[FY] Freyd P., Yetter D.: Coherence theorems via knot theory. J. Pure Appl. Algebra 78 (1992), no. 1, 49–76.

[G] Gukov,S: Three-Dimensional Quantum Gravity, Chern-Simons Theory, and the A-Polynomial, hep-th/0306165

[GI] Garcia-Islas M.: $(2+1)$-dimensional quantum gravity, spin networks and asymptotics, Classical Quantum Gravity 21 (2004), no. 2, 445–464

[GMS] Gel'fand I. M., Minlos R.A., Shapiro Z.Ya.: Representations of the Rotation and Lorentz groups and their Applications, Oxford Pergamon, 1963

[GGV] Gel'fand I. M., Graev M. I., Vilenkin N. Ya.:Generalized Functions, volume 5, Integral geometry and representation Theory, Academic Press 1966

[GN] Garoufalidis S, Bar-Natan D: On the Melvin-Morton-Rozansky Conjecture Inventiones Mathematicae 125 (1996) pp 103-133

[H] Habiro, K.: On the quantum sl_2 invariants of knots and integral homology spheres. Invariants of knots and 3-manifolds (Kyoto, 2001), 55–68 (electronic), Geom. Topol. Monogr., 4, Geom. Topol. Publ., Coventry, 2002.

[K] Kassel C. : Quantum Groups, Springer-Werlag 1994

[Kir] Kirillov A. A.: Elements of the Theory of Representations, Springer-Verlag, 1976.

[Ko] Kontsevich M: Vassiliev's knot invariants. I. M. Gel'fand Seminar, 137–150, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993.

[KS] Klimyk A., Schmudgen K.: Quantum Groups and their representations, Springer, 1997

[KT] Kashaev R.M., Tirkkonen O.: A Proof of the Volume Conjecture for Torus Knots, J. Math. Sci. (New York) **115** (2003), no. 1, 2033 -2036

[L] Lang S.: $SL(2;\mathbb{R})$, Addison-Wesley, Reading, MA, 1975

[LM1] Le TQ, Murakami J., The Universal Vassiliev-Kontsevitch Invariant for Framed Oriented Links, Comp. Math. 102, 1996, 41-64

[LM2] Le T Q, Murakami J.: Representations of the Category of Tangles by Kontsevich's Iterated Integral, Commun. Math. Phys., **168** (1995) 535-562

[LM3] Le T Q, Murakami J.: Kontsevich's integral for the Kauffman polynomial. Nagoya Math. J. 142 (1996), 39–65.

[LNT] Le T Q, Bar-Natan D., Thurston D. : Two Applications of Elementary Knot Theory to Lie Algebras and Vassiliev Invariants Geometry and Topology 7-1 (2003) pp 1-31

[M] Majid S.: Foundations of quantum group theory. Cambridge University Press, Cambridge, 1995

[MM] Melvin P.M., Morton H.R.: The Colored Jones Function, Commun Math Phys 169 (3), pp 501-520 May 1995

[Mor] Morton, H. R. The coloured Jones function and Alexander polynomial for torus knots. Math. Proc. Cambridge Philos. Soc. 117 (1995), no. 1, 129–135.

[NR] Noui, K., Roche, Ph.: Cosmological Deformation of Lorentzian Spin Foam Models, Class.Quant.Grav. 20 (2003) 3175-3214, gr-qc/0211109

[P] Pusz W.: Irreducible unitary representations of quantum Lorentz group. Commun. Math. Phys, 152,591-626 (1993)

[PS] Pflaum M., Schottenloher M.: Holomorphic Deformations of Hopf Algebras and Aplications to Quantum Groups, Journal of Geometry and Physics **28** (1998) 31-44

[PW] Poddles P., Woronowicz S.L.: A Quantum Deformation of the Lorentz Group, Commun. Math. Phys,130,381-431 (1990)

[R1] Rozansky, L. Higher order terms in the Melvin-Morton expansion of the colored Jones polynomial. Comm. Math. Phys. 183 (1997), no. 2, 291–306.

[R2] Rozansky, L. The universal R-matrix, Burau representation, and the Melvin-Morton expansion of the colored Jones polynomial. Adv. Math. 134 (1998), no. 1, 1–31.

[RT] Reshetikhin N. Yu.; Turaev V. G: Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys. **127** (1990), no. 1, 1–26.

[SS] Sternin B., Shalatov E.: Borel-Laplace Transform and Asymptotic Theory, CrC Press (1996)

[T] Takeuchi M.: Finite-dimensional representations of the quantum Lorentz group. Comm. Math. Phys. **144** (1992), no. 3, 557–580.

[VAR] Varadarajan, V. S.: Lie groups, Lie algebras, and their representations, Prentice-Hall, 1974

[W] Witten E.: Quantization of Chern-Simons gauge theory with complex gauge group. Comm. Math. Phys. **137** (1991), no. 1, 29–66.

[Wi1] Willerton, S.: The Kontsevich integral and algebraic structures on the space of diagrams, Knots in Hellas '98 (Delphi), 530-546, Ser. Knots Everything, **24**, World Sci. Publishing, River Edge, NJ, 2000.

[Wi2] Willerton S.: Vassiliev invariants and the Hopf algebra of chord diagrams, Math. Proc. Cambridge Philos. Soc. **119** (1996), no. 1, 55-65