Invariants of knots, loop braids and knotted surfaces derived from finite 2-groups

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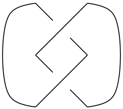
Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- Asphericity means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a non-splittable link.

E.g. L = 

Definition: (n=type) Let $n \in \mathbb{Z}_0^+$.

An n-type is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\textbf{-types}\}$ be the category with objects the n-types.

Given n-types X and Y,
morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon\ \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

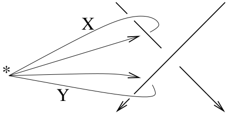
1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Two pointed maps $f, f'\colon X \to Y$ are pointed homotopic iff the induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

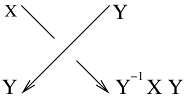
In particular it follows that:

Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: Wirtinger presentation for $\pi_1(S^3 \setminus K)$.

A generator for each arc of projection. A relation for each crossing:





Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical. (Likely it never is.)

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at the homotopy 2-type $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... To be explained later.

We will see 2-groups as being represented by crossed modules.

Crossed modules

Definition (Crossed module)

A crossed module $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ is given by:

- A group map $\partial \colon E \to G$.
  (G is called the "base-group". E is the "principal group".)
- A left action $\triangleright$ of G on E, by automorphisms.
- Such that the following conditions (Peiffer equations) hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- Let V be a set, G a group. Consider a map $\partial_0 \colon V \to G$.
  We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \left(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Facts about crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

1. Crossed modules and their maps form a category.
2. Each crossed module embeds into an exact sequence like:

   $$\pi_2(\mathcal{G}) \doteq \ker(\partial) \xrightarrow{i} \boxed{E \xrightarrow{\partial} G} \xrightarrow{p} \pi_1(\mathcal{G}) \doteq \operatorname{coker}(\partial).$$
3. Yield cohomology class $\omega \in H^3(\pi_1(\mathcal{G}), \pi_2(\mathcal{G}))$, the k-invariant.
4. An algebraic 2-type is a triple (A, K, ω), where A is an abelian group with a left action of K, and $\omega \in H^3(K, A)$.

   We have a fundamental algebraic 2-type functor:

   $$\{\textbf{Pointed topological spaces}\} \to \{\text{Algebraic 2-types}\}$$

   sending a space X to the triple $(\pi_2(X), \pi_1(X), k(X))$, called the algebraic 2-type of X.

   We also have a functor:

   $$\rho_2\colon\ \{\textbf{Crossed Modules}\} \to \{\text{Algebraic 2-types}\}$$
   $$\mathcal{G} \mapsto (\pi_2(\mathcal{G}), \pi_1(\mathcal{G}), k(\mathcal{G})).$$

The algebraic 2-type of a space classifies its homotopy 2-type. But non pointed-homotopic maps between 2-types may induce the same map on fundamental algebraic 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s \colon G \to E'$.

We have category $\{\textbf{Cof-Crossed Modules}\}/\cong$.

Objects crossed modules $\mathcal{G} = (\partial \colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

*Ho(***Crossed Modules***)* $\cong$ **2-types***. I.e.*

$\{\textbf{Cof-Crossed Modules}\}/\cong$ *is equivalent to category of 2-types.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem *Ho*(**Crossed Modules**) $\cong$ **2-types**. I.e.

$\{$**Cof-Crossed Modules**$\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{$**CW-complexes**$\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps.

  We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 0 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X
If X and Y are homotopic CW-complexes then $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "$=$" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module.
Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\#E)^{\text{number of 1-cells of X}}} \# \operatorname{hom}(\Pi_2(X, X^1), \mathcal{G}),$$

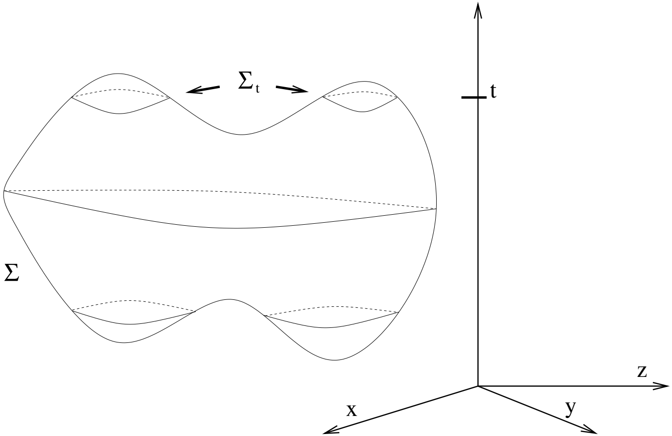
does not depend on the chosen CW-decomposition of X.
Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.
Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

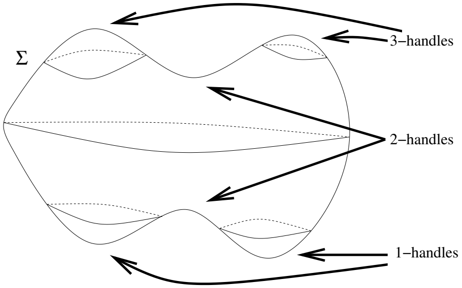
$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".



Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

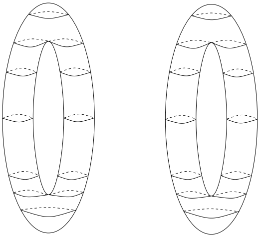


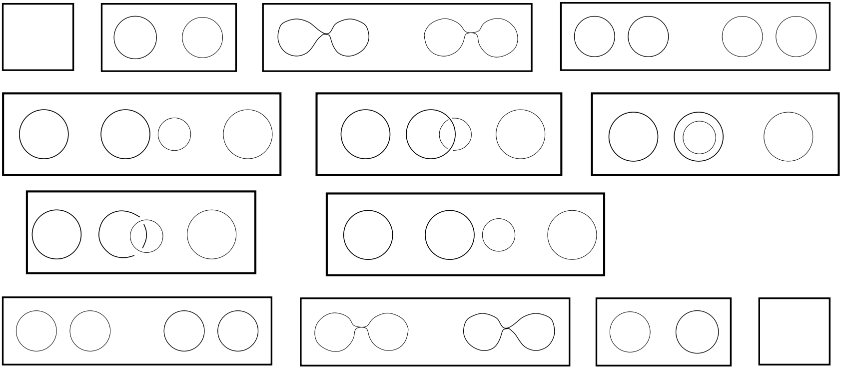
Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

A movie for a knotted union Σ of two tori

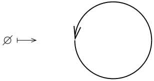




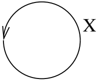
Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

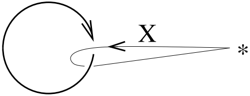
Locally, an oriented minimal point looks like:



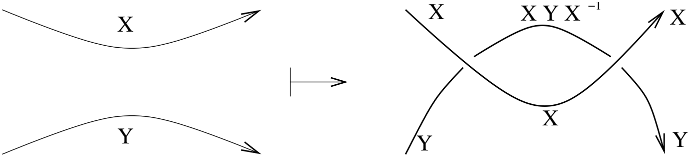
A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:



Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

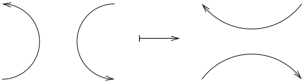


As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

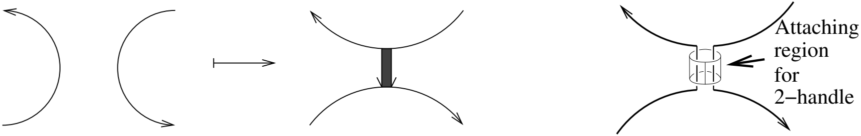


Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

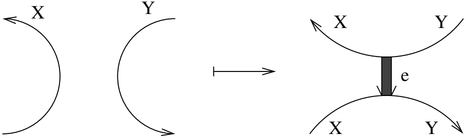
Locally, an (oriented) saddle point looks like:



When passing saddle point, add a 'band', kept throughout movie:
This band will later bookkeep where the saddle point was made,
and the attaching region of corresponding 2-handle of M.



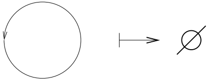
Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

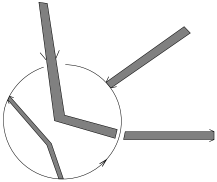


$$\partial(e) = X^{-1}Y.$$

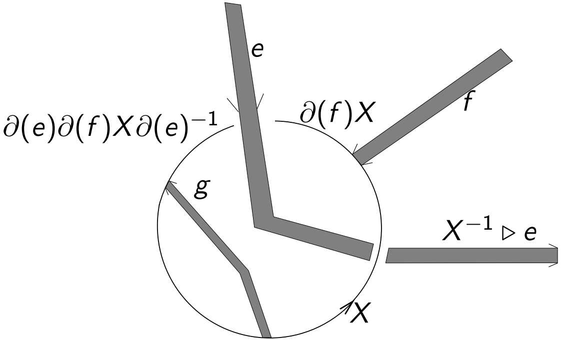
Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Maximal points

Locally, an oriented maximal point looks like: 

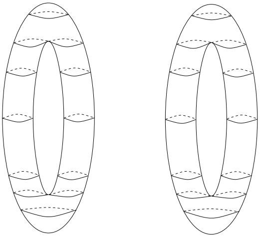
Some bands will possibly be present.
Before maximal point, configuration looks like: 

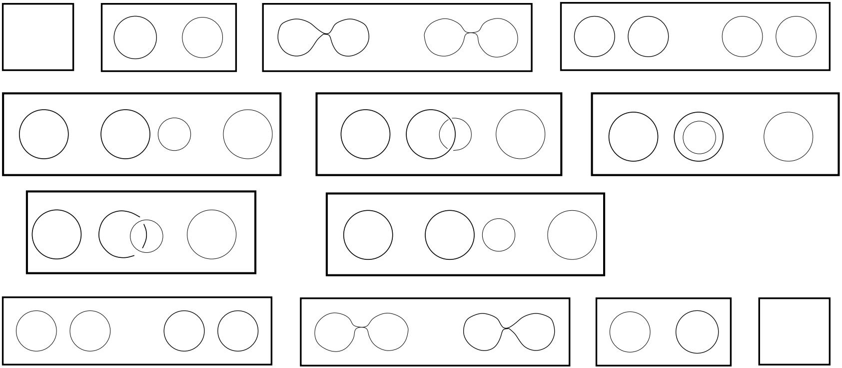
In this case the 2-relations are as below:



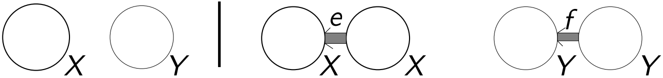
2-relation:
$e\ f\ (X^{-1} \triangleright e^{-1})$
$= 1.$

A movie for a knotted union Σ of two tori



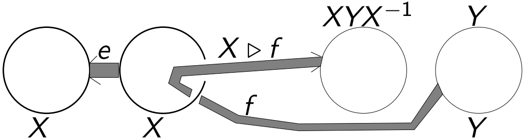


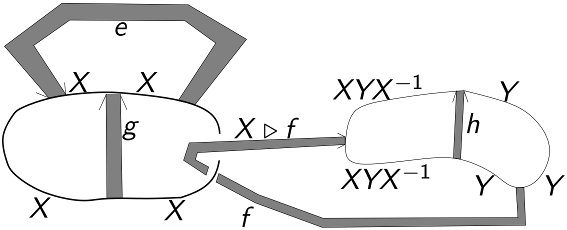
Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



$\partial(e) = 1$
$\partial(f) = 1$

$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.





$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\,e^{-1}\,(X \triangleright f^{-1})\;f = 1$
$(X \triangleright f)f^{-1} = 1.$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto 1 \\ h \mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

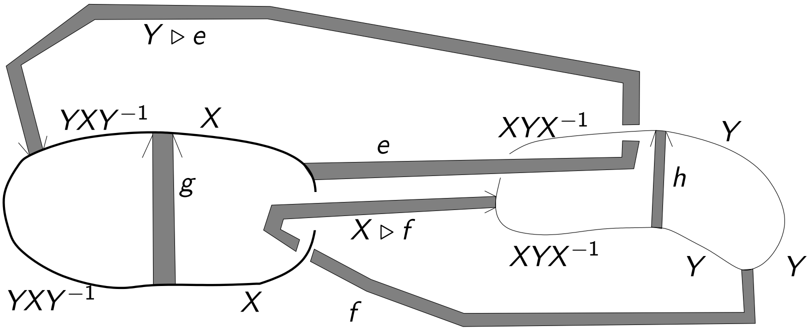
$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\} / < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\big\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\big\}(\#E).$$

Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:



$\partial(e) = 1$
$\partial(f) = 1$
$\partial(g) = YXY^{-1}X^{-1}$
$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\; f = 1$

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto [Y,X] \\ h \mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y \triangleright e)\, e^{-1} \\ (X \triangleright f^{-1})\ f \\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f)\ + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\left\{(X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY=YX, \\ (Y \triangleright e) - e - (X \triangleright f)\ + f = 0}\right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

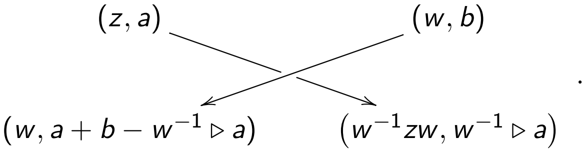
More results on $I_{\mathcal{G}}(S^4 \setminus \Sigma)$

Let $\mathcal{G} = (E \to G, \triangleright)$ be a finite crossed module.

1. $\Sigma \mapsto I_{\mathcal{G}}(S^4 \setminus \Sigma)$ is able to separate between pairs of knotted surfaces with different knot groups. (For some choices of $\mathcal{G}$.)

2. Recal Shin Satoh's “tube-map”
$$T\colon \{\text{Welded links}\} \to \{\text{Knotted Tori}\}$$

   Suppose $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$.
   The welded knot invariant
$$K \mapsto I_{\mathcal{G}}(S^4 \setminus T(K))$$
   can be calculated from the biquandle with set $G \times E$:



Applications to 3+1D topological phases of matter start here....

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