Crossed modules, homotopy 2-types, knotted surfaces and welded knots

Topological Quantum Field Theory Club (IST, Lisbon)

30th October 2020

João Faria Martins (University of Leeds)

LEVERHULME TRUST

UNIVERSITY OF LEEDS

Partially funded by the Leverhulme Trust research project grant:
RPG-2018-029: "Emergent Physics From Lattice Models of Higher Gauge Theory"

Thanks due to Tim Porter, Paul Martin, Alex Bullivant, Celeste Damiani, Gustavo Granja, Louis Kauffman.

Other Refs. on homotopy 2-types of 2knot complements

- *S. J. Lomonaco*: The homotopy groups of knots. I: How to compute the algebraic 2-type. Pac. J. Math. 95, 349–390 (1981).
- *A. I. Suciu*: Infinitely many ribbon knots with the same fundamental group. Math. Proc. Camb. Philos. Soc. 98, 481–492 (1985).
- *S. P. Plotnick* and *A. I. Suciu*: k-invariants of knotted 2-spheres. Comment. Math. Helv. 60, 54–84 (1985).

Other Refs. on homotopy 2-types of 2knot complements

- *S. J. Lomonaco*: The homotopy groups of knots. I: How to compute the algebraic 2-type. Pac. J. Math. 95, 349–390 (1981).
- *A. I. Suciu*: Infinitely many ribbon knots with the same fundamental group. Math. Proc. Camb. Philos. Soc. 98, 481–492 (1985).
- *S. P. Plotnick* and *A. I. Suciu*: k-invariants of knotted 2-spheres. Comment. Math. Helv. 60, 54–84 (1985).

Other Refs. on homotopy 2-types of 2knot complements

- *S. J. Lomonaco*: The homotopy groups of knots. I: How to compute the algebraic 2-type. Pac. J. Math. 95, 349–390 (1981).
- *A. I. Suciu*: Infinitely many ribbon knots with the same fundamental group. Math. Proc. Camb. Philos. Soc. 98, 481–492 (1985).
- *S. P. Plotnick* and *A. I. Suciu*: k-invariants of knotted 2-spheres. Comment. Math. Helv. 60, 54–84 (1985).

Other Refs. on homotopy 2-types of 2knot complements

- *S. J. Lomonaco*: The homotopy groups of knots. I: How to compute the algebraic 2-type. Pac. J. Math. 95, 349–390 (1981).
- *A. I. Suciu*: Infinitely many ribbon knots with the same fundamental group. Math. Proc. Camb. Philos. Soc. 98, 481–492 (1985).
- *S. P. Plotnick* and *A. I. Suciu*: k-invariants of knotted 2-spheres. Comment. Math. Helv. 60, 54–84 (1985).

Other Refs. on homotopy 2-types of 2knot complements

- *S. J. Lomonaco*: The homotopy groups of knots. I: How to compute the algebraic 2-type. Pac. J. Math. 95, 349–390 (1981).
- *A. I. Suciu*: Infinitely many ribbon knots with the same fundamental group. Math. Proc. Camb. Philos. Soc. 98, 481–492 (1985).
- *S. P. Plotnick* and *A. I. Suciu*: k-invariants of knotted 2-spheres. Comment. Math. Helv. 60, 54–84 (1985).

Main refs. for this talk:

- JFM.: The Fundamental Crossed Module of the Complement of a Knotted Surface. Transactions of the American Mathematical Society. 361 (2009), 4593-4630.
- JFM, Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Compositio Mathematica. Volume 144, Issue 04, July 2008.
- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. Advances in Theoretical and Mathematical Physics Volume 23 (2019).
- Damiani C, JFM, Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898

Main refs. for this talk:

- JFM.: The Fundamental Crossed Module of the Complement of a Knotted Surface. Transactions of the American Mathematical Society. 361 (2009), 4593-4630.

- JFM, Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Compositio Mathematica. Volume 144, Issue 04, July 2008.

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. Advances in Theoretical and Mathematical Physics Volume 23 (2019).

- Damiani C, JFM, Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898

Main refs. for this talk:

- JFM.: The Fundamental Crossed Module of the Complement of a Knotted Surface. Transactions of the American Mathematical Society. 361 (2009), 4593-4630.

- JFM, Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Compositio Mathematica. Volume 144, Issue 04, July 2008.

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. Advances in Theoretical and Mathematical Physics Volume 23 (2019).

- Damiani C, JFM, Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898

Main refs. for this talk:

- JFM.: The Fundamental Crossed Module of the Complement of a Knotted Surface. Transactions of the American Mathematical Society. 361 (2009), 4593-4630.

- JFM, Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Compositio Mathematica. Volume 144, Issue 04, July 2008.

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. Advances in Theoretical and Mathematical Physics Volume 23 (2019).

- Damiani C, JFM, Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898

Main refs. for this talk:

- JFM.: The Fundamental Crossed Module of the Complement of a Knotted Surface. Transactions of the American Mathematical Society. 361 (2009), 4593-4630.
- JFM, Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Compositio Mathematica. Volume 144, Issue 04, July 2008.
- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. Advances in Theoretical and Mathematical Physics Volume 23 (2019).
- Damiani C, JFM, Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898

Some references on combinatorial homotopy

- *R. Brown, P. Higgins, R Sivera*: Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids. With contributions by Christopher D. Wensley and Sergei V. Soloviev. Zurich: European Mathematical Society (EMS) (2011)
- *H. J. Baues*: Combinatorial homotopy and 4-dimensional complexes. Berlin etc.: Walter de Gruyter (1991)
- *H. J. Baues*: Algebraic homotopy. Cambridge etc.: Cambridge University Press (1989)
- *J. H. C. Whitehead*: Combinatorial homotopy. I. and II. Bull. Am. Math. Soc. 55. (1949)
- *J. C. Baez* and *A. D. Lauda*: Higher-dimensional algebra. V: 2-Groups. Theory Appl. Categ. 12, 423–491 (2004).

Some references on combinatorial homotopy

- *R. Brown, P. Higgins, R Sivera*: Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids. With contributions by Christopher D. Wensley and Sergei V. Soloviev. Zurich: European Mathematical Society (EMS) (2011)
- *H. J. Baues*: Combinatorial homotopy and 4-dimensional complexes. Berlin etc.: Walter de Gruyter (1991)
- *H. J. Baues*: Algebraic homotopy. Cambridge etc.: Cambridge University Press (1989)
- *J. H. C. Whitehead*: Combinatorial homotopy. I. and II. Bull. Am. Math. Soc. 55. (1949)
- *J. C. Baez* and *A. D. Lauda*: Higher-dimensional algebra. V: 2-Groups. Theory Appl. Categ. 12, 423–491 (2004).

Some references on combinatorial homotopy

- *R. Brown, P. Higgins, R Sivera*: Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids. With contributions by Christopher D. Wensley and Sergei V. Soloviev. Zurich: European Mathematical Society (EMS) (2011)
- *H. J. Baues*: Combinatorial homotopy and 4-dimensional complexes. Berlin etc.: Walter de Gruyter (1991)
- *H. J. Baues*: Algebraic homotopy. Cambridge etc.: Cambridge University Press (1989)
- *J. H. C. Whitehead*: Combinatorial homotopy. I. and II. Bull. Am. Math. Soc. 55. (1949)
- *J. C. Baez* and *A. D. Lauda*: Higher-dimensional algebra. V: 2-Groups. Theory Appl. Categ. 12, 423–491 (2004).

Some references on combinatorial homotopy

- *R. Brown, P. Higgins, R Sivera*: Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids. With contributions by Christopher D. Wensley and Sergei V. Soloviev. Zurich: European Mathematical Society (EMS) (2011)
- *H. J. Baues*: Combinatorial homotopy and 4-dimensional complexes. Berlin etc.: Walter de Gruyter (1991)
- *H. J. Baues*: Algebraic homotopy. Cambridge etc.: Cambridge University Press (1989)
- *J. H. C. Whitehead*: Combinatorial homotopy. I. and II. Bull. Am. Math. Soc. 55. (1949)
- *J. C. Baez* and *A. D. Lauda*: Higher-dimensional algebra. V: 2-Groups. Theory Appl. Categ. 12, 423–491 (2004).

Some references on combinatorial homotopy

- *R. Brown, P. Higgins, R Sivera*: Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids. With contributions by Christopher D. Wensley and Sergei V. Soloviev. Zurich: European Mathematical Society (EMS) (2011)
- *H. J. Baues*: Combinatorial homotopy and 4-dimensional complexes. Berlin etc.: Walter de Gruyter (1991)
- *H. J. Baues*: Algebraic homotopy. Cambridge etc.: Cambridge University Press (1989)
- *J. H. C. Whitehead*: Combinatorial homotopy. I. and II. Bull. Am. Math. Soc. 55. (1949)
- *J. C. Baez* and *A. D. Lauda*: Higher-dimensional algebra. V: 2-Groups. Theory Appl. Categ. 12, 423–491 (2004).

Some references on combinatorial homotopy

- *R. Brown, P. Higgins, R Sivera*: Nonabelian algebraic topology. Filtered spaces, crossed complexes, cubical homotopy groupoids. With contributions by Christopher D. Wensley and Sergei V. Soloviev. Zurich: European Mathematical Society (EMS) (2011)

- *H. J. Baues*: Combinatorial homotopy and 4-dimensional complexes. Berlin etc.: Walter de Gruyter (1991)

- *H. J. Baues*: Algebraic homotopy. Cambridge etc.: Cambridge University Press (1989)

- *J. H. C. Whitehead*: Combinatorial homotopy. I. and II. Bull. Am. Math. Soc. 55. (1949)

- *J. C. Baez* and *A. D. Lauda*: Higher-dimensional algebra. V: 2-Groups. Theory Appl. Categ. 12, 423–491 (2004).

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\textbf{-types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

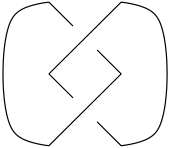
Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L =



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequently omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

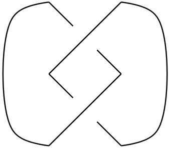
morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L =



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequently omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

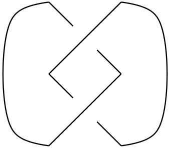
Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L = 

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

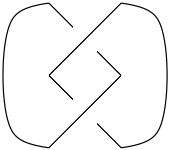
morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L =



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

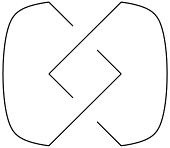
morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L =



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell.
   (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\textbf{-types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

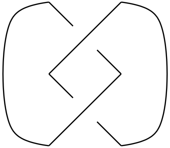
morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L =



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

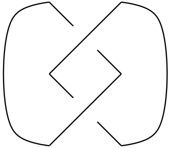
morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L =



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

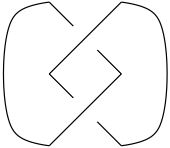
Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L = 

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\textbf{-types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

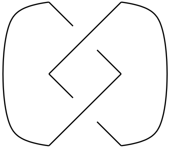
Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L =



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\textbf{-types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

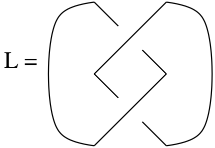
morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1 \colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f' \colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_* \colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1 \colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f' \colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_* \colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1 \colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f' \colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_* \colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1 \colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f' \colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_* \colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{\textbf{1-types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{\text{1-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:

Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.

A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \ \{\text{1-}\textbf{types}\} \to \{\textbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1 \colon \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f' \colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_* \colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon\ \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

$$\pi_1\colon \{\text{1-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:

1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

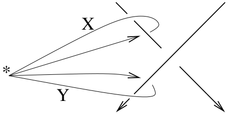
$$\pi_1\colon \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:



1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

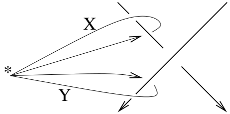
$$\pi_1\colon\ \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:



1-types and knot complements

Therefore, complements of non-splittable links in S^3 are 1-types.

Well known theorem: The fundamental group functor

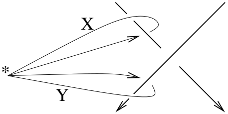
$$\pi_1\colon\ \{1\text{-}\mathbf{types}\} \to \{\mathbf{groups}\}$$

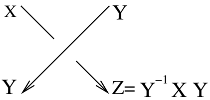
is an equivalence of categories. This implies:

1. Two 1-types X and Y are homotopic iff $\pi_1(X) \cong \pi_1(Y)$.
2. Maps $f, f'\colon X \to Y$, of 1-types, are pointed homotopic iff induced maps $f_*, f'_*\colon \pi_1(X) \to \pi_1(Y)$ are equal.

In particular, combining with Papakyriakopoulos theorem, we have:
Theorem: The homotopy type of the complement of a non-splittable link $L \subset S^3$ is classified by $\pi_1(S^3 \setminus L)$.

Also recall: *Wirtinger presentation* for $\pi_1(S^3 \setminus K)$.
A generator for each arc of projection. A relation for each crossing:





Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.

(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

.... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... To be explained later.

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... To be explained later.

We will see 2-groups as being represented by crossed modules.

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... To be explained later.

We will see 2-groups as being represented by crossed modules.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}$, where $g \in G, e \in E$;
  2. $\partial(e) \triangleright f = efe^{-1}$, where $e, f \in E$.

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial \colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$

  2. $\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial \colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial \colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}$, where $g \in G, e \in E$;
  2. $\partial(e) \triangleright f = efe^{-1}$, where $e, f \in E$.

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}$, where $g \in G, e \in E$;
  2. $\partial(e) \triangleright f = efe^{-1}$, where $e, f \in E$.

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a\in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}$, where $g \in G, e \in E$;
  2. $\partial(e) \triangleright f = efe^{-1}$, where $e, f \in E$.

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a\in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}$, where $g \in G, e \in E$;
  2. $\partial(e) \triangleright f = efe^{-1}$, where $e, f \in E$.

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a\in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial \colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial \colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial \colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a\in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a\in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

Crossed modules

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
  (G is called the *"base-group"*. E is the *"principal group"*.)
- A left action $\triangleright$ of G on E, by automorphisms,
- such that the following conditions *(Peiffer equations)* hold:
  1. $\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$
  2. $\partial(e) \triangleright f = efe^{-1}, \text{ where } e, f \in E.$

Example

- G a group; A an abelian group.
  Consider a left-action $\triangleright$ of G on A, by automorphisms.
  We have a crossed module $\mathcal{G} = (A \xrightarrow{a \in A \longmapsto 1_G} G, \triangleright)$.
- $\partial\colon A \to G$, map of abelian groups. Action $g \triangleright_{trivial} a = a$.
  Then $\mathcal{G} = (\partial\colon A \to G, \triangleright_{trivial})$ is a crossed module.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial \colon E \to G, \triangleright)$

A group map $\partial \colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad} \colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad} \colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial \colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$.
  $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial \colon E \to G, \triangleright)$

A group map $\partial \colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad} \colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad} \colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial \colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial}).$

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.

More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

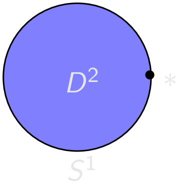
$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.



More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

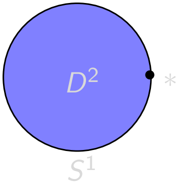
$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.



More examples of crossed modules $\mathcal{G} = (\partial\colon E \to G, \triangleright)$

A group map $\partial\colon E \to G$. A left action $\triangleright$ of G on E. With

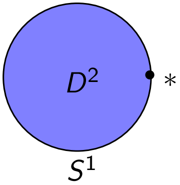
$$\partial(g \triangleright e) = g\partial(e)g^{-1}, \text{ where } g \in G, e \in E;$$
$$\partial(e) \triangleright f = efe^{-1}, \quad \text{where } e, f \in E.$$

- Let H be any group. $G = \mathrm{Aut}(H)$. $\partial = \mathrm{Ad}\colon H \to \mathrm{Aut}(H)$. $(\mathrm{Ad}\colon H \to \mathrm{Aut}(H), \triangleright)$ is a crossed module.

- Let $(M, N, *)$ be a pair of spaces. We have a crossed module:

$$\Pi_2(M, N, *) = (\partial\colon \pi_2(M, N, *) \to \pi_1(N, *), \triangleright).$$

- $\Pi_2(D^2, S^1, *) = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$.



Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \left(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \left(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$. We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \left(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$.

We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \left(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \left(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$. We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \big(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\big).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0\colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0\colon V \to G\rangle = \left(\partial\colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \left(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0\colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0\colon V \to G\rangle = \left(\partial\colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0\colon V \to G$. We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0\colon V \to G\rangle = \big(\partial\colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\big).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0\colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0\colon V \to G\rangle = \big(\partial\colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\big).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0\colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0\colon V \to G\rangle = \big(\partial\colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\big).$$

Universal property

such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \big(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\big).$$

Universal property

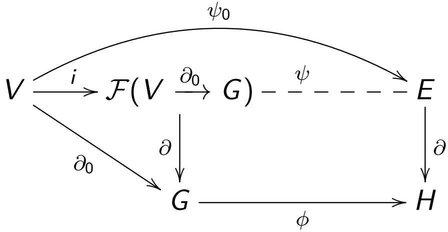
such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0\colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0\colon V \to G\rangle = \big(\partial\colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\big).$$

Universal property



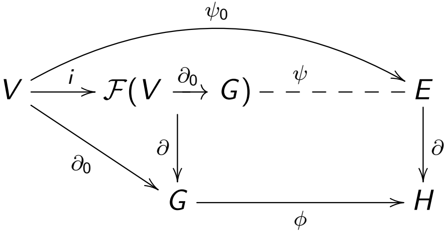
such that we have a crossed module map:

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$. We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = \big(\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\big).$$

Universal property



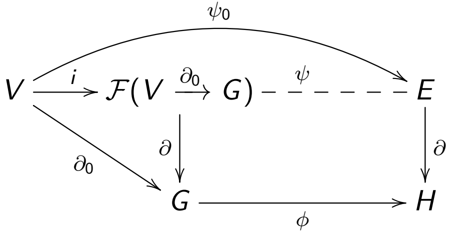
such that we have a crossed module map:

Free crossed modules

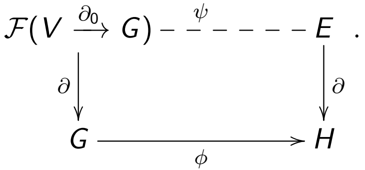
Let V be a set, G a group. Consider a set map $\partial_0\colon V \to G$. We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0\colon V \to G\rangle = \left(\partial\colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright\right).$$

Universal property



such that we have a crossed module map:

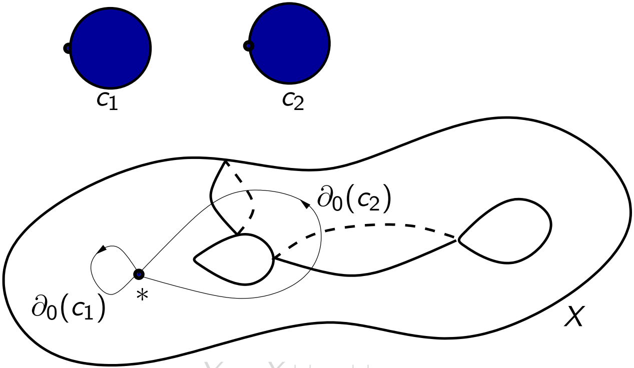


Free crossed modules and Whitehead theorem

$$Y = X \cup c_1 \cup c_2.$$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2-cells\} \xrightarrow{\partial_0} \pi_1(X)$.

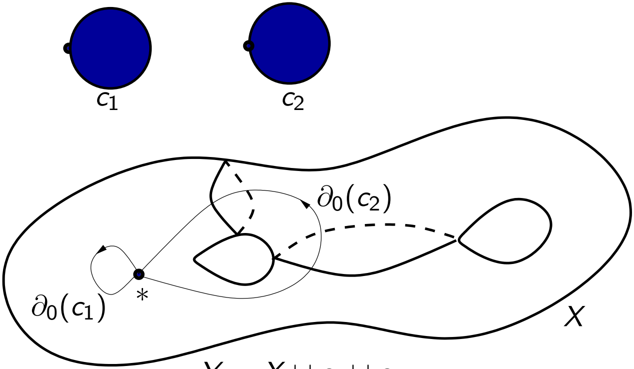
Free crossed modules and Whitehead theorem



$Y = X \cup c_1 \cup c_2.$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2 - cells\} \xrightarrow{\partial_0} \pi_1(X)$.

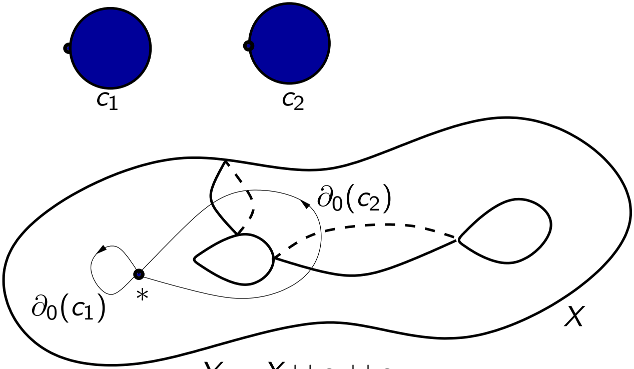
Free crossed modules and Whitehead theorem



$$Y = X \cup c_1 \cup c_2.$$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2 - cells\} \xrightarrow{\partial_0} \pi_1(X)$.

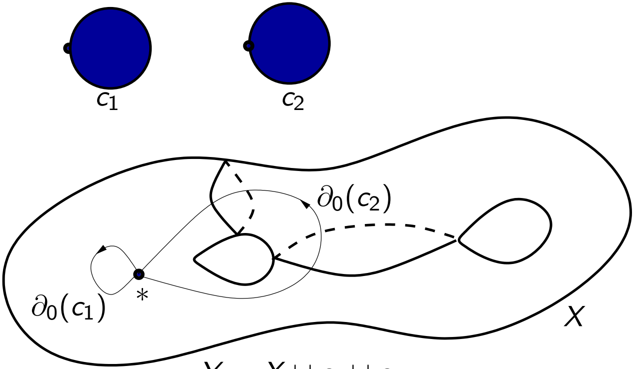
Free crossed modules and Whitehead theorem



$Y = X \cup c_1 \cup c_2.$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2 - cells\} \xrightarrow{\partial_0} \pi_1(X)$.

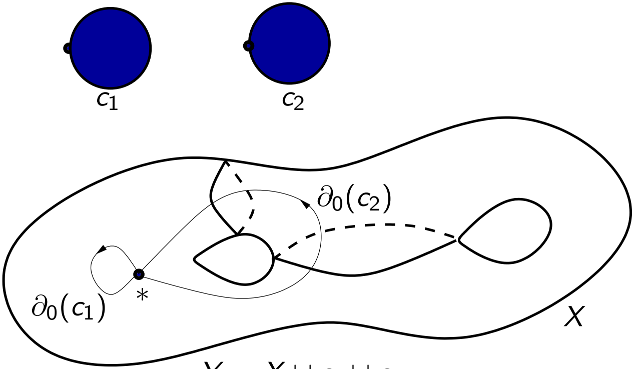
Free crossed modules and Whitehead theorem



$Y = X \cup c_1 \cup c_2.$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2 - cells\} \xrightarrow{\partial_0} \pi_1(X)$.

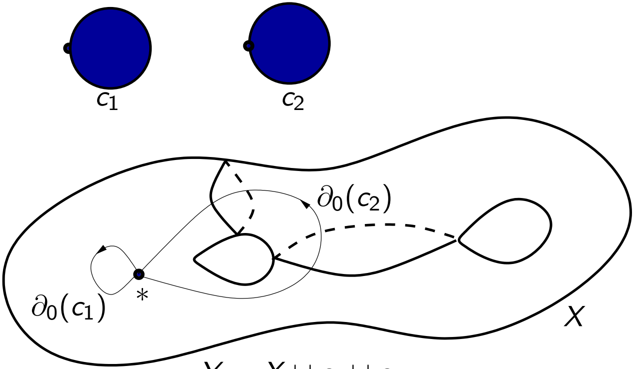
Free crossed modules and Whitehead theorem



$Y = X \cup c_1 \cup c_2.$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2 - cells\} \xrightarrow{\partial_0} \pi_1(X)$.

Free crossed modules and Whitehead theorem



$Y = X \cup c_1 \cup c_2.$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2 - cells\} \xrightarrow{\partial_0} \pi_1(X)$.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem
Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s \colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial \colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s \colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial \colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

*Ho($\{$**Crossed Modules**$\}$) is equivalent to $\{$**2-types**$\}$.*

I.e.:

*the category $\{$**Cof-Crossed Modules**$\}/\cong$*

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s \colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial \colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

*Ho($\{$**Crossed Modules**$\}$) is equivalent to $\{$**2-types**$\}$.*

I.e.:

*the category $\{$**Cof-Crossed Modules**$\}/\cong$*

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$*.*

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

*Ho($\{$**Crossed Modules**$\}$) is equivalent to $\{$**2-types**$\}$.*

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

Homotopy of crossed modules

A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s\colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\cong$.
Objects are crossed modules $\mathcal{G} = (\partial\colon E \to F)$; F a free group.
Maps $\mathcal{G} \to \mathcal{G}'$ are homotopy classes of maps $\mathcal{G} \to \mathcal{G}'$.

Theorem

Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$*.*

I.e.:

the category $\{$**Cof-Crossed Modules**$\}/\cong$

is equivalent to category of 2-types.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps.
  We have a functor
  $\Pi_2\colon \{\textbf{CW-complexes}\} /\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types,* Π_2 *is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of* X. *Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{$**Crossed Modules**$\}) \cong \{$**2-types**$\}$. I.e. $\{$**Cof-Crossed Modules**$\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{$**CW-complexes**$\}/\cong$ be the category with objects reduced CW-complexes, <u>with chosen CW-decomposition</u>. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \; \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e.

$\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps.

  We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\}\ /\cong\ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X.* *Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{$**Crossed Modules**$\}) \cong \{$**2-types**$\}$. I.e. $\{$**Cof-Crossed Modules**$\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial \colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{$**CW-complexes**$\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps.

  We have a functor

$$\Pi_2 \colon \{\textbf{CW-complexes}\} / \cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial \colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps.

  We have a functor

$$\Pi_2 \colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem *Ho*({**Crossed Modules**}) $\cong$ {**2-types**} . I.e.
{**Cof-Crossed Modules**}$/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let {**CW-complexes**}$/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\}\ /\cong\ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\}\ /\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e.
$\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, <u>with chosen CW-decomposition</u>. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial \colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2 \colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. *$\Pi_2(X, X^1)$ faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\}\ /\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X.* *Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e.
$\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps.
  We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\}\ /\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. *$\Pi_2(X, X^1)$ faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps.

  We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\}\ /\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X.* *Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \operatorname{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{$**Crossed Modules**$\}) \cong \{$**2-types**$\}$. I.e.
$\{$**Cof-Crossed Modules**$\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{$**CW-complexes**$\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. *$\Pi_2(X, X^1)$ faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e.
$\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\} \ / \cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. *$\Pi_2(X, X^1)$ faithfully represents the homotopy 2-type of X.* *Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\} \,/\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e.
$\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\} \,/\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e. $\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps. We have a functor

$$\Pi_2\colon \{\textbf{CW-complexes}\}/\cong \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence* $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.

The fundamental crossed module $\Pi_2(X, X^1)$

Theorem $Ho(\{\textbf{Crossed Modules}\}) \cong \{\textbf{2-types}\}$. I.e.
$\{\textbf{Cof-Crossed Modules}\}/\cong$ is equivalent to category of 2-types.

This equivalence of categories can be made more concrete.

- Given a reduced CW-complex X, let X^1 be its one-skeleton. We have a crossed module:

$$\Pi_2(X, X^1) = (\partial\colon \pi_2(X, X^1) \to \pi_1(X^1), \triangleright).$$

- Let $\{\textbf{CW-complexes}\}/\cong$ be the category with objects reduced CW-complexes, with chosen CW-decomposition. Maps $X \to Y$ are pointed homotopy classes of pointed maps.
  We have a functor

$$\Pi_2\colon \ \{\textbf{CW-complexes}\}\ /\cong \ \longrightarrow \{\textbf{Cof-Crossed Modules}\}/\cong.$$

Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
2. $\Pi_2(X, X^1)$ *faithfully represents the homotopy 2-type of X. Hence $\pi_2(X) = \ker(\partial)$, $\pi_1(X) = \mathrm{coker}(\partial)$, $k(X) = k(\Pi_2(X))$.*

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
   is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
   is obtained from the free crossed module $\Pi_2(X^2, X^1)$
   by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}$(1-cells): free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
   is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
   is obtained from the free crossed module $\Pi_2(X^2, X^1)$
   by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.

Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}$(1-cells): free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial \colon \pi_2(X^2, X^1) \to \pi_1(X^1))$ is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U} \left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial \colon \pi_2(X^3, X^1) \to \pi_1(X^1))$ is obtained from the free crossed module $\Pi_2(X^2, X^1)$ by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U} \left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}$(1-cells): free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
   is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
   is obtained from the free crossed module $\Pi_2(X^2, X^1)$
   by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}$(1-cells): free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
   is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
   is obtained from the free crossed module $\Pi_2(X^2, X^1)$
   by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
   is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
   is obtained from the free crossed module $\Pi_2(X^2, X^1)$
   by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \;\middle|\; \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
   is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
   is obtained from the free crossed module $\Pi_2(X^2, X^1)$
   by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
   is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
   is obtained from the free crossed module $\Pi_2(X^2, X^1)$
   by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial \colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial \colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial \colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle .$$

3. $\Pi_2(X, X^1) = (\partial \colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle .$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

Presentation of $\Pi_2(X, X^1)$ by generators and relations

Let X be a reduced CW-complex. X^i union of cells of index $\leq i$.
Procedure to describe a presentation of the crossed module:

$$\Pi_2(X, X^1) = (\pi_2(X, X^1) \to \pi_1(X^1))$$

by generators and relations. (In the world of crossed modules.)

1. $\pi_1(X^1) = \mathcal{F}(\text{1-cells})$: free group on the set of 1-cells of X.
2. $\Pi_2(X^2, X^1) = (\partial\colon \pi_2(X^2, X^1) \to \pi_1(X^1))$
is the free crossed module on the attaching maps of the 2-cells.

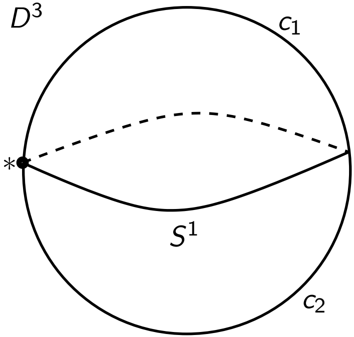
$$\Pi_2(X^2, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \right\rangle.$$

3. $\Pi_2(X, X^1) = (\partial\colon \pi_2(X^3, X^1) \to \pi_1(X^1))$
is obtained from the free crossed module $\Pi_2(X^2, X^1)$
by imposing a crossed module 2-relation for each 3-cell.

$$\Pi_2(X, X^1) = \mathcal{U}\left\langle \{\text{2-cells}\} \xrightarrow{\partial} \pi_1(X^1) \mid \partial(c) = 1 \text{ for each } c \in \{\text{3-cells}\} \right\rangle.$$

Also Π_2 satisfies a van Kampen type property. (Brown-Higgins).

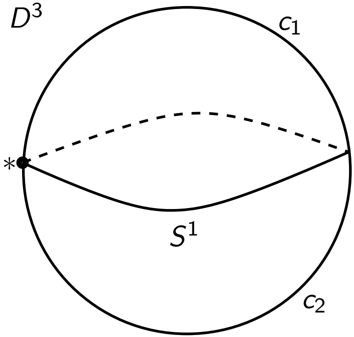
Presentation of $\Pi_2(D^3, S^1)$ by generators and relations



$$\Pi_2(S^2, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow[c_2 \mapsto 1]{c_1 \mapsto 1} (\mathbb{Z}, +) \right\rangle = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}, \triangleright_{trivial})$$

$$\Pi_2(D^3, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow[c_2 \mapsto 1]{c_1 \mapsto 1} (\mathbb{Z}, +) \mid c_1 = c_2 \right\rangle = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$$

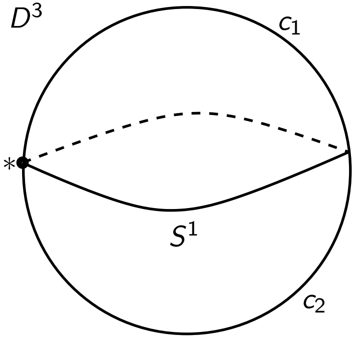
Presentation of $\Pi_2(D^3, S^1)$ by generators and relations



$$\Pi_2(S^2, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \right\rangle = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}, \triangleright_{trivial})$$

$$\Pi_2(D^3, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \mid c_1 = c_2 \right\rangle = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$$

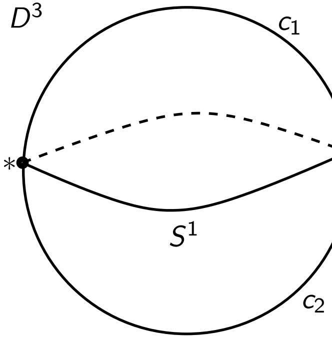
Presentation of $\Pi_2(D^3, S^1)$ by generators and relations



$$\Pi_2(S^2, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \right\rangle = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}, \triangleright_{trivial})$$

$$\Pi_2(D^3, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \mid c_1 = c_2 \right\rangle = (\mathbb{Z} \xrightarrow{\mathrm{id}} \mathbb{Z}, \triangleright_{trivial})$$

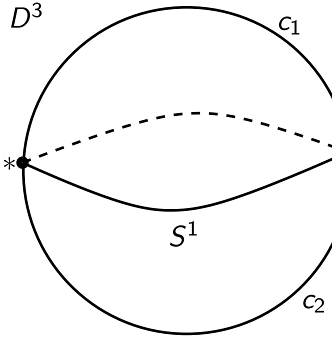
Presentation of $\Pi_2(D^3, S^1)$ by generators and relations



$$\Pi_2(S^2, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow[]{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \right\rangle = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}, \triangleright_{trivial})$$

$$\Pi_2(D^3, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow[]{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \mid c_1 = c_2 \right\rangle = (\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}, \triangleright_{trivial})$$

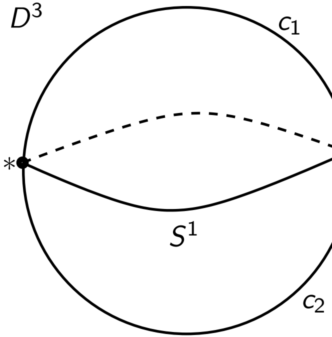
Presentation of $\Pi_2(D^3, S^1)$ by generators and relations



$$\Pi_2(S^2, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow[c_2 \mapsto 1]{c_1 \mapsto 1} (\mathbb{Z}, +) \right\rangle = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}, \triangleright_{trivial})$$

$$\Pi_2(D^3, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow[c_2 \mapsto 1]{c_1 \mapsto 1} (\mathbb{Z}, +) \mid c_1 = c_2 \right\rangle = (\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}, \triangleright_{trivial})$$

Presentation of $\Pi_2(D^3, S^1)$ by generators and relations



$$\Pi_2(S^2, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow[]{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \right\rangle = (\mathbb{Z} \oplus \mathbb{Z} \xrightarrow{(a,b) \mapsto a+b} \mathbb{Z}, \triangleright_{trivial})$$

$$\Pi_2(D^3, S^1) = \mathcal{U}\left\langle \{c_1, c_2\} \xrightarrow[]{\substack{c_1 \mapsto 1 \\ c_2 \mapsto 1}} (\mathbb{Z}, +) \mid c_1 = c_2 \right\rangle = (\mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z}, \triangleright_{trivial})$$

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes.

.

Proposition Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \operatorname{hom}(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes.

Proposition Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.
Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.
Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X
Let X and Y be homotopic CW-complexes.

$$\Pi_2(X, X^1) \qquad = \Pi_2(Y, Y^1) \qquad .$$

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module.
Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.
Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.
Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \qquad\qquad = \Pi_2(Y, Y^1) \qquad\qquad .$$

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad\qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \qquad\qquad = \Pi_2(Y, Y^1) \qquad\qquad .$$

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad\qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1) \quad = \Pi_2(Y, Y^1) \qquad .$$

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \operatorname{hom}(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \qquad .$$

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X. Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}).$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module.

Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad\qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X
Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.
Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.
Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X
Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.
Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.
Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\; m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \qquad \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X. Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\# E)^{\text{number of 1-cells of X}}} \# \operatorname{hom}(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.
Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.
Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "$=$" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\#E)^{\text{number of 1-cells of X}}} \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "$=$" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\# E)^{\text{number of 1-cells of X}}} \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X. Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\#E)^{\text{number of 1-cells of X}}} \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\# E)^{\text{number of 1-cells of X}}} \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\# E)^{\text{number of 1-cells of X}}} \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\# \pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\#E)^{\text{number of 1-cells of X}}} \# \operatorname{hom}(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X

Let X and Y be homotopic CW-complexes. $\exists\ m, n \in \mathbb{Z}_0^+$ such that:

$$\Pi_2(X, X^1) \vee \Pi_2(D^2, S^1)^{\vee m} = \Pi_2(Y, Y^1) \vee \Pi_2(D^2, S^1)^{\vee n}.$$

We are using "=" to say "isomorphic'.

Proposition Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module. Let X be a finite reduced CW-complex. The quantity:

$$I_{\mathcal{G}}(X) = \frac{1}{(\#E)^{\text{number of 1-cells of X}}} \# \hom(\Pi_2(X, X^1), \mathcal{G}),$$

does not depend on the chosen CW-decomposition of X.

Moreover, $I_{\mathcal{G}}(X)$ is a homotopy invariant of X.

Interpretation:

$$I_{\mathcal{G}}(X) = \sum_{f \in \pi_0(\mathrm{TOP}(X, B_{\mathcal{G}}))} \frac{1}{\#\pi_1(\mathrm{TOP}(X, B_{\mathcal{G}}), f)}$$

$B_{\mathcal{G}}$ is the classifying space of $\mathcal{G}$. $\mathrm{TOP}(X, B_{\mathcal{G}})$ function space.

Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".

Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.

(Any genus, any number of components.)

Suppose the projection on the t-variable is a Morse function in Σ.

To simplify, suppose critical points appear in increasing order.

Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".

Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".

Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

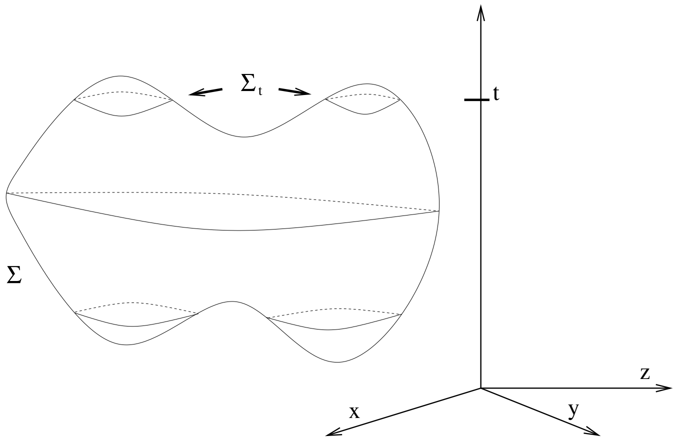
Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".

Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".

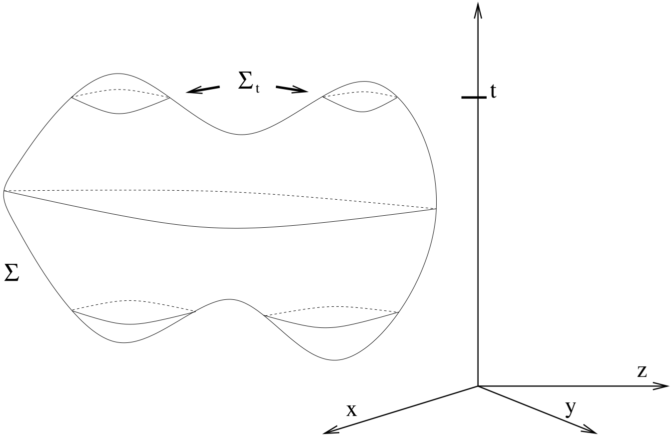
Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".



Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".



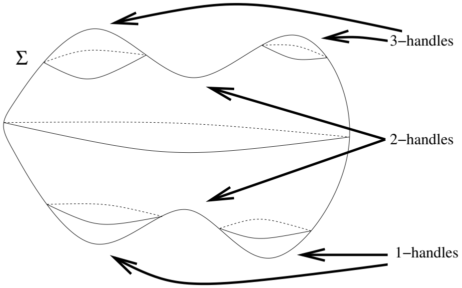
Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

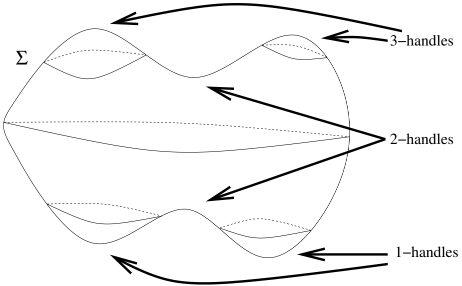


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$. (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$. (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$. (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

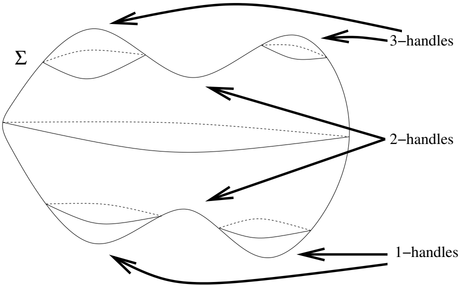


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

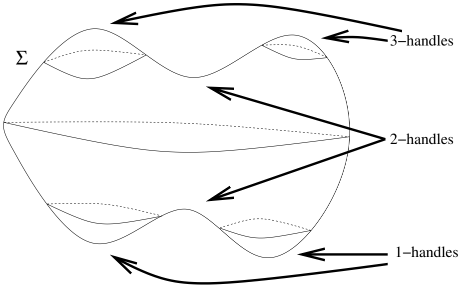


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

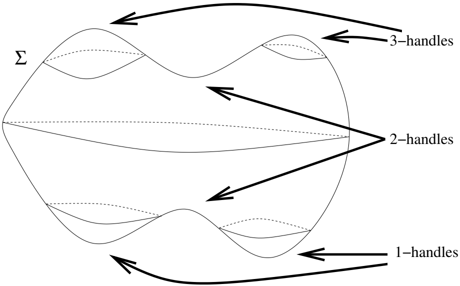


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

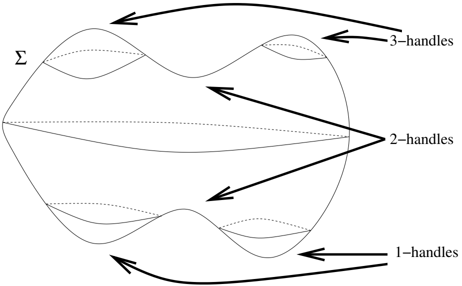


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$. (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$. (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$. (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

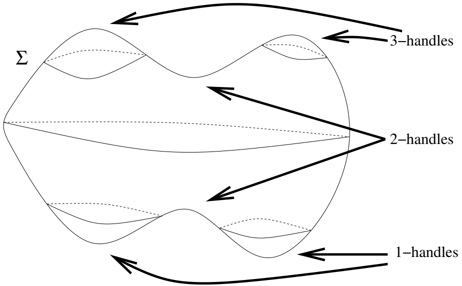


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

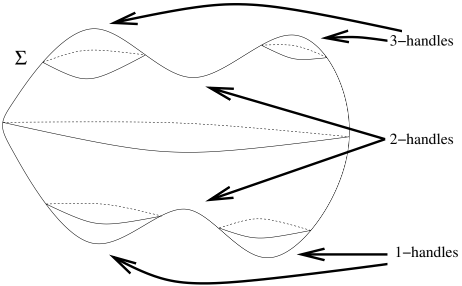


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

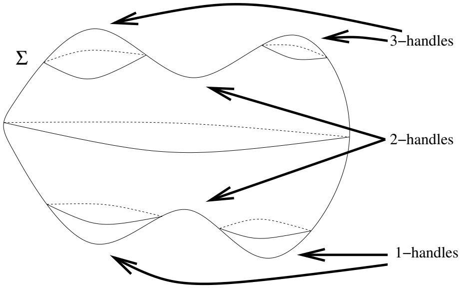


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

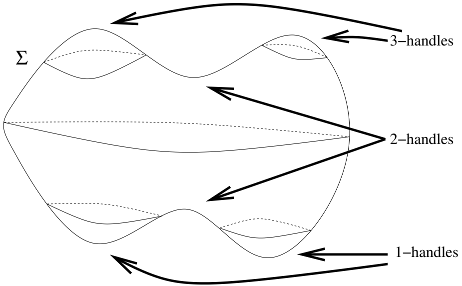


Let $M^{(i)}$ be union of handles of index $\leq i$.

- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$. (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$. (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$. (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

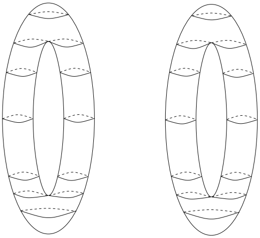


Let $M^{(i)}$ be union of handles of index $\leq i$.

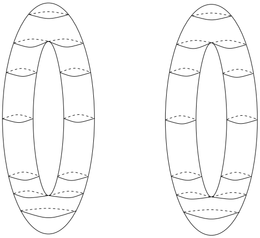
- A minimal point in Σ yields a 1-handle of $S^4 \setminus \Sigma$.
  (Hence a free generator of the group $\pi_1(M^{(1)})$.)
- A saddle point in Σ yields a 2-handle of $S^4 \setminus \Sigma$.
  (Hence a free crossed module generator of $\Pi_2(M^{(2)}, M^{(1)})$.)
- A maximal point in Σ yields a 3-handle of $S^4 \setminus \Sigma$.
  (Hence a 2-relation needs to be imposed on $\Pi_2(M^{(2)}, M^{(1)})$ in order to get to $\Pi_2(M, M^{(1)})$.)

A presentation for $\Pi_2(M, M^{(1)})$ can be derived from a 'movie' of Σ.

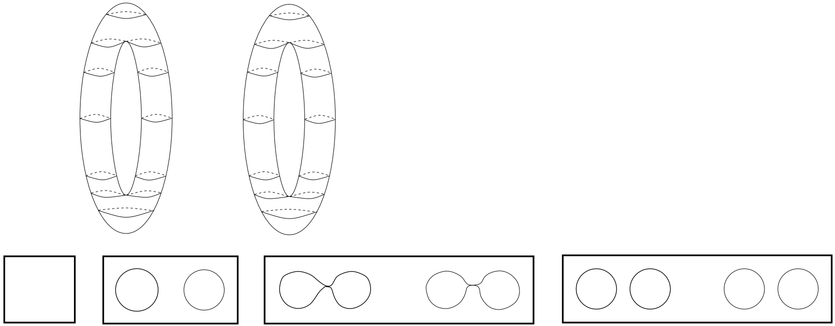
A movie for a knotted union Σ of two tori



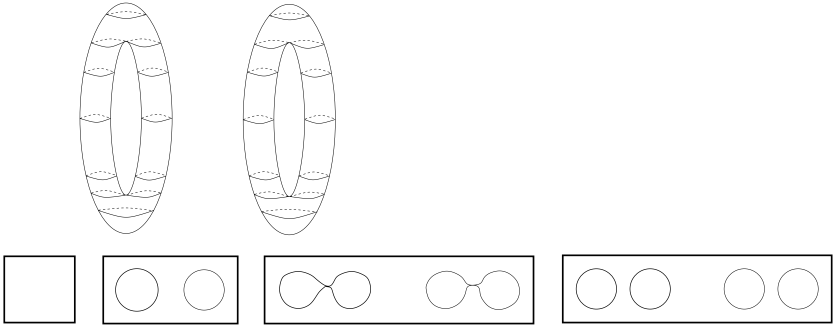
A movie for a knotted union Σ of two tori



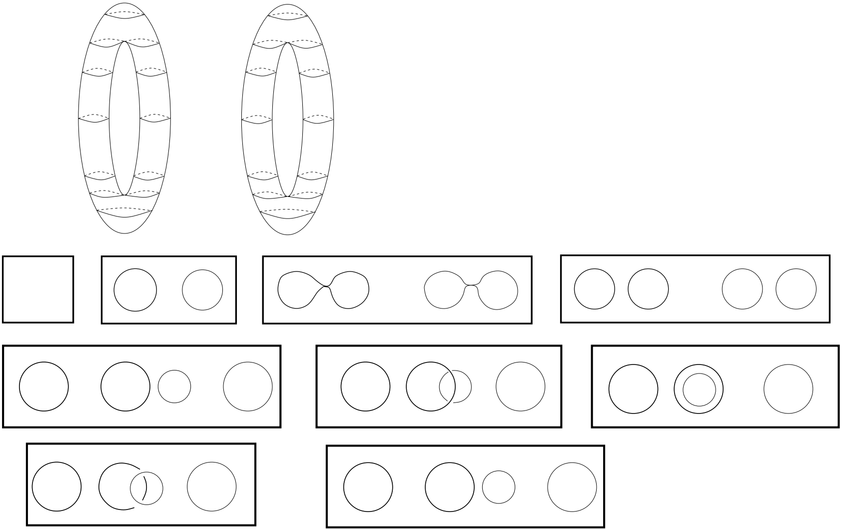
A movie for a knotted union Σ of two tori



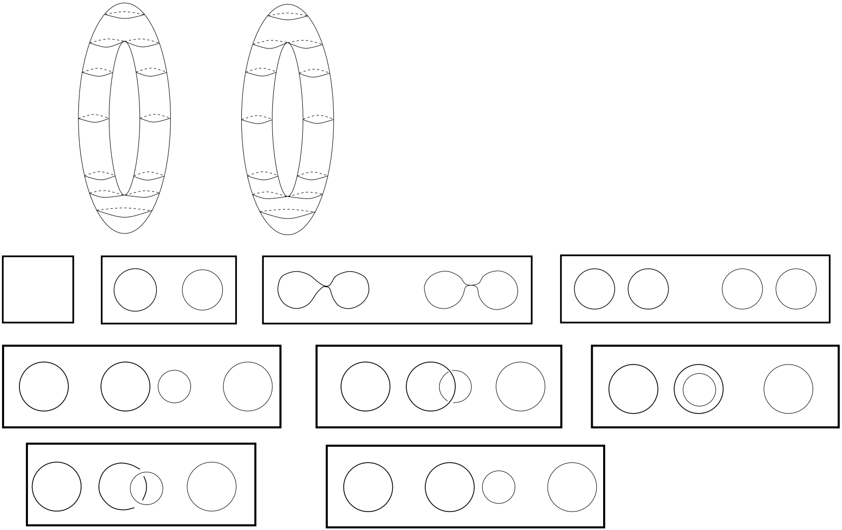
A movie for a knotted union Σ of two tori



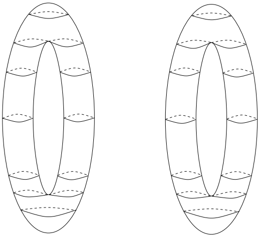
A movie for a knotted union Σ of two tori

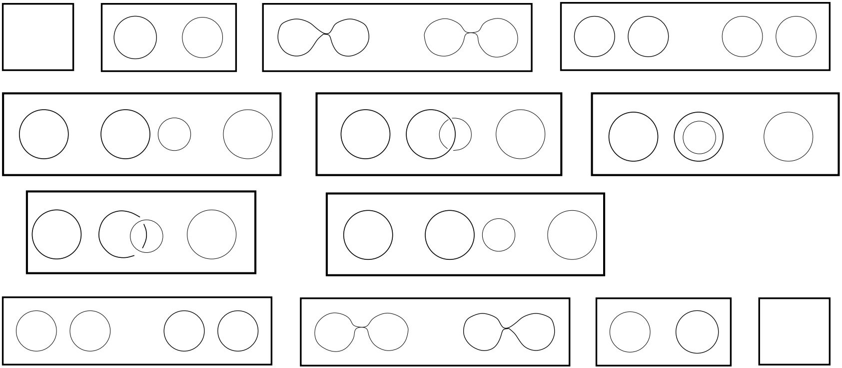


A movie for a knotted union Σ of two tori



A movie for a knotted union Σ of two tori





Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

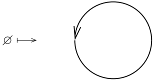
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent. There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:



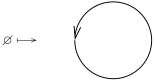
A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like: 

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

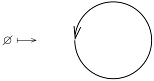
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:



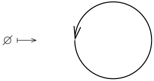
A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like: 

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

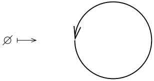
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

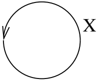
Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:



A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:



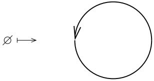
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent. There are relations between generators at different times. For $R2$:

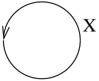
Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:



A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:



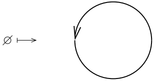
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

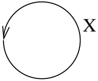
Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

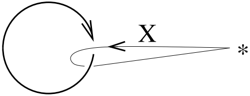
Locally, an oriented minimal point looks like:



A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:



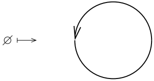
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

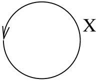


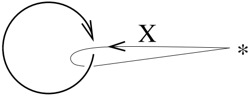
As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like: 

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it: 

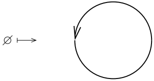
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as: 

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

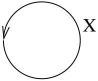
Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

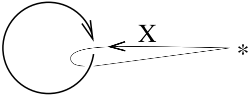
Locally, an oriented minimal point looks like:



A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:



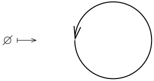
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

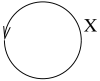


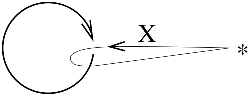
As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like: 

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it: 

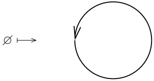
Concretely, $X \in \pi_1(M^{(1)})$ can be defined as: 

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

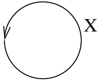
Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

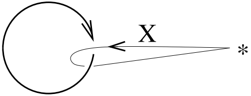
Locally, an oriented minimal point looks like:



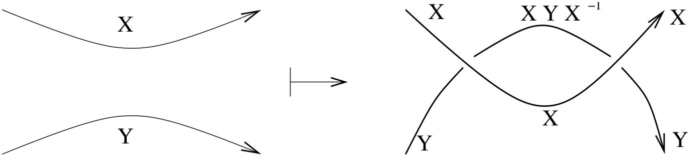
A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:



Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:



As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:



Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:

When passing saddle point, add a 'band', kept throughout movie:
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.

Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:

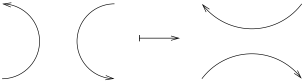
When passing saddle point, add a ‘band’, kept throughout movie:
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.

Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:



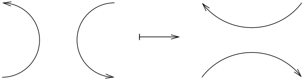
When passing saddle point, add a 'band', kept throughout movie:
This band will later bookkeep where the saddle point was made,
and the attaching region of corresponding 2-handle of M.

Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:



When passing saddle point, add a ‘band’, kept throughout movie:

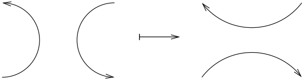
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.

Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

Bands are to be kept and evolve throughout the rest of the movie. Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

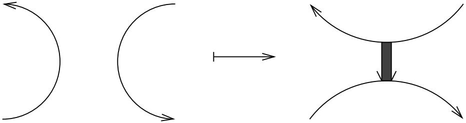
Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:



When passing saddle point, add a 'band', kept throughout movie:

This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.

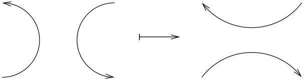


Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

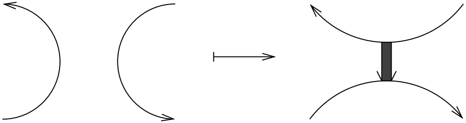
Bands are to be kept and evolve throughout the rest of the movie. Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:



When passing saddle point, add a 'band', kept throughout movie:
This band will later bookkeep where the saddle point was made,
and the attaching region of corresponding 2-handle of M.

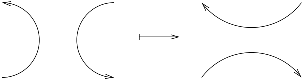


Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

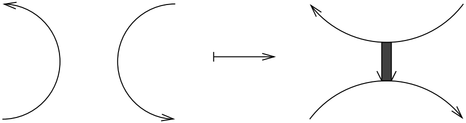
Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:



When passing saddle point, add a ‘band’, kept throughout movie:
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.

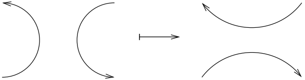


Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

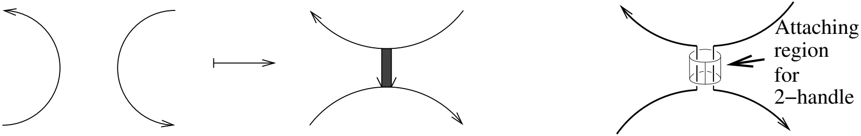
Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:



When passing saddle point, add a 'band', kept throughout movie:
This band will later bookkeep where the saddle point was made,
and the attaching region of corresponding 2-handle of M.

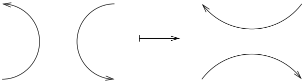


Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

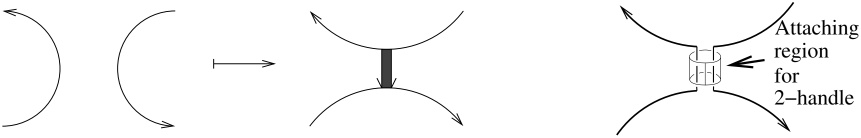
Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:



When passing saddle point, add a ‘band’, kept throughout movie:
This band will later bookkeep where the saddle point was made,
and the attaching region of corresponding 2-handle of M.

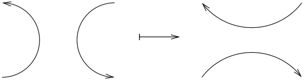


Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

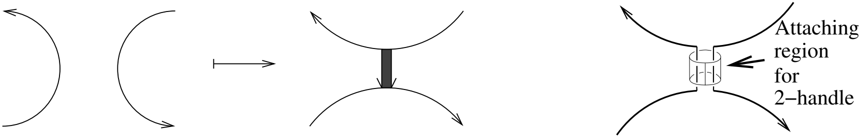
Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

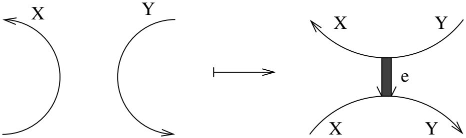
Locally, an (oriented) saddle point looks like:



When passing saddle point, add a ‘band’, kept throughout movie:
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.



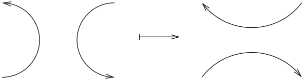
Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.



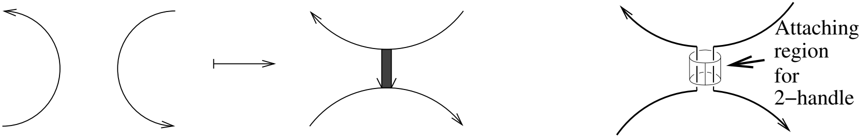
Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

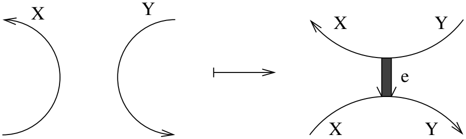
Locally, an (oriented) saddle point looks like:



When passing saddle point, add a ‘band’, kept throughout movie:
This band will later bookkeep where the saddle point was made,
and the attaching region of corresponding 2-handle of M.



Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

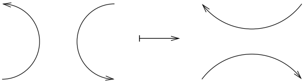


$$\partial(e) = X^{-1}Y.$$

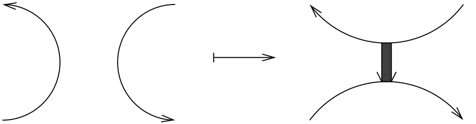
Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

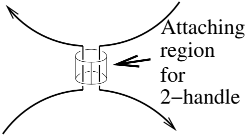
Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:

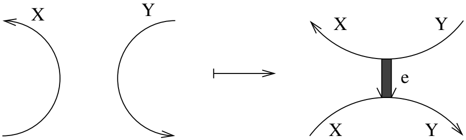


When passing saddle point, add a ‘band’, kept throughout movie:
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.





Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

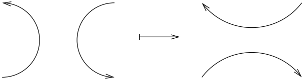


$$\partial(e) = X^{-1}Y.$$

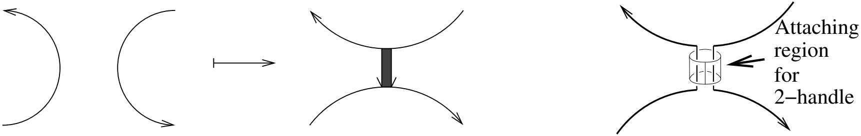
Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

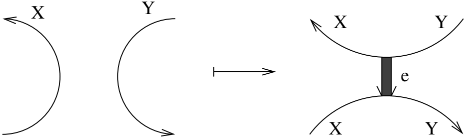
Locally, an (oriented) saddle point looks like:



When passing saddle point, add a 'band', kept throughout movie:
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.



Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.



$$\partial(e) = X^{-1}Y.$$

Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

Maximal points

Locally, an oriented maximal point looks like:

Some bands will possibly be present.
Before maximal point, configuration looks like:

In this case the 2-relations are as below:

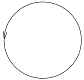
Maximal points

Locally, an oriented maximal point looks like:

Some bands will possibly be present.
Before maximal point, configuration looks like:

In this case the 2-relations are as below:

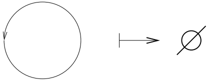
Maximal points

Locally, an oriented maximal point looks like:  $\longmapsto \emptyset$

Some bands will possibly be present.
Before maximal point, configuration looks like:

In this case the 2-relations are as below:

Maximal points

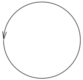
Locally, an oriented maximal point looks like: 

Some bands will possibly be present.

Before maximal point, configuration looks like:

In this case the 2-relations are as below:

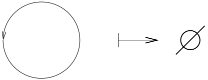
Maximal points

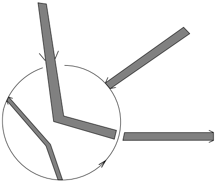
Locally, an oriented maximal point looks like:  $\longmapsto \varnothing$

Some bands will possibly be present.
Before maximal point, configuration looks like:

In this case the 2-relations are as below:

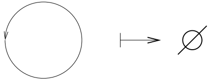
Maximal points

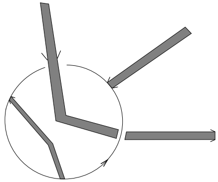
Locally, an oriented maximal point looks like: 

Some bands will possibly be present.
Before maximal point, configuration looks like: 

In this case the 2-relations are as below:

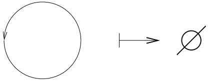
Maximal points

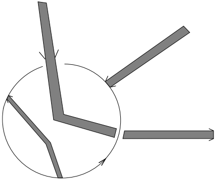
Locally, an oriented maximal point looks like: 

Some bands will possibly be present.
Before maximal point, configuration looks like: 

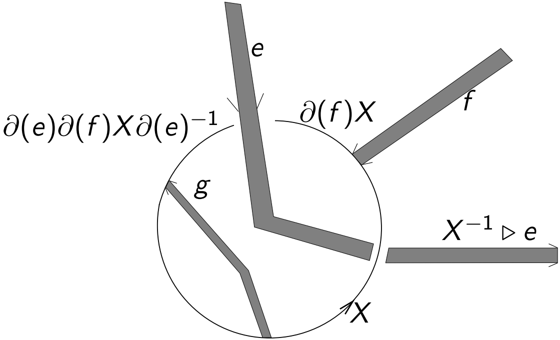
In this case the 2-relations are as below:

Maximal points

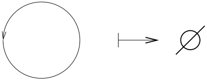
Locally, an oriented maximal point looks like: 

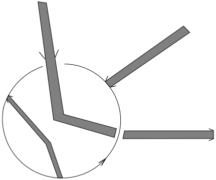
Some bands will possibly be present.
Before maximal point, configuration looks like: 

In this case the 2-relations are as below:

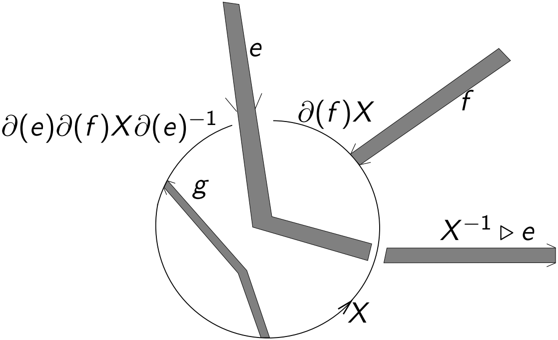


Maximal points

Locally, an oriented maximal point looks like: 

Some bands will possibly be present.
Before maximal point, configuration looks like: 

In this case the 2-relations are as below:

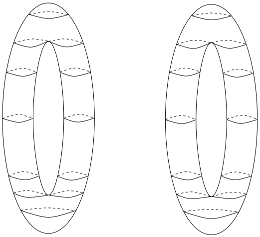


2-relation:

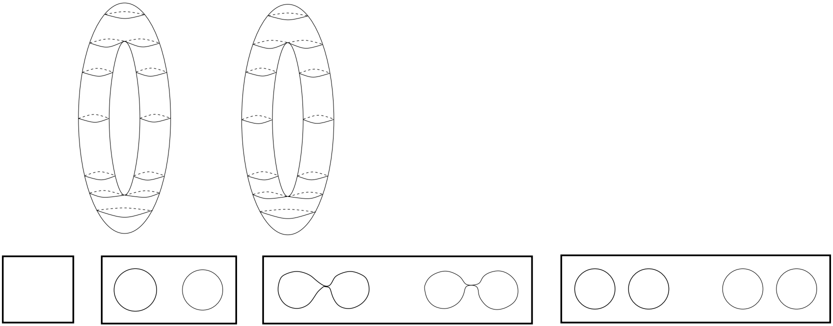
$$e \ f \ (X^{-1} \triangleright e^{-1}) = 1$$

A movie for a knotted union Σ of two tori

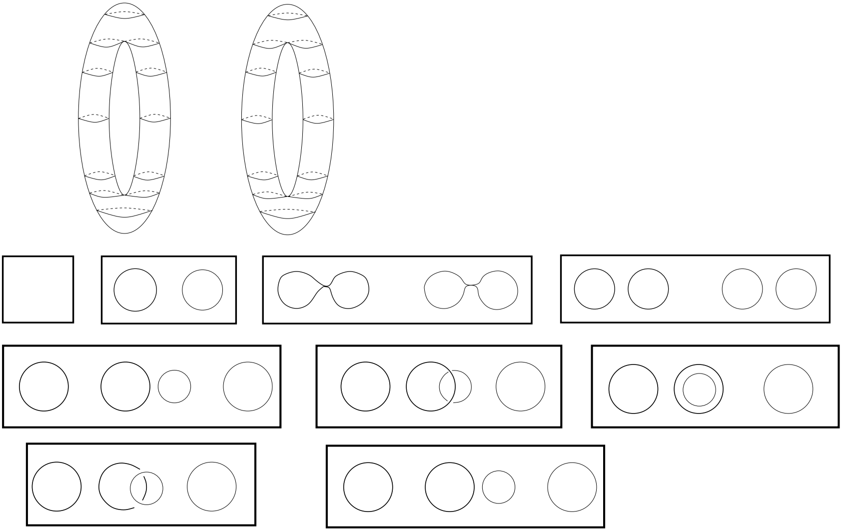
A movie for a knotted union Σ of two tori



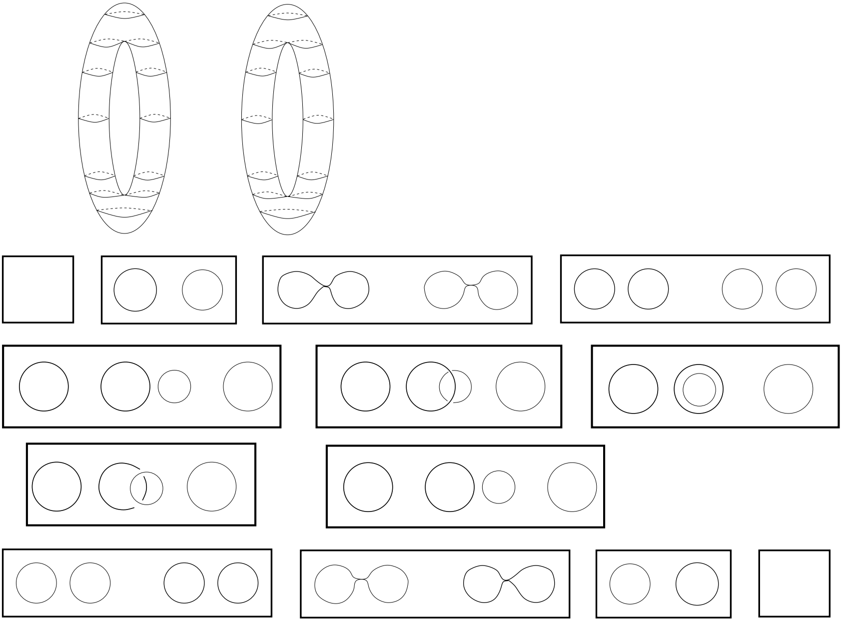
A movie for a knotted union Σ of two tori



A movie for a knotted union Σ of two tori



A movie for a knotted union Σ of two tori

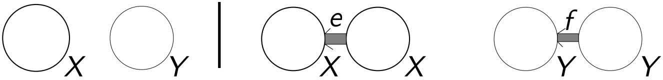


Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise

$\partial(e) = 1$
$\partial(f) = 1$

$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$
$(X \triangleright f) f^{-1} = 1.$

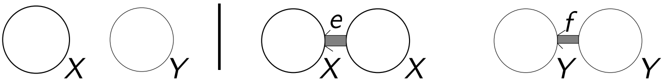
$\Sigma =$ Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



$\partial(e) = 1$
$\partial(f) = 1$

$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$
$(X \triangleright f) f^{-1} = 1.$

Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise

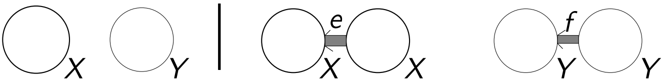


$\partial(e) = 1$
$\partial(f) = 1$

$X, Y \in \pi_1(M^{(1)})$;

$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$
$(X \triangleright f)f^{-1} = 1.$

Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise

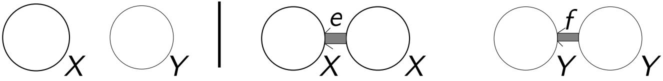


$\partial(e) = 1$
$\partial(f) = 1$

$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.

$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$
$(X \triangleright f) f^{-1} = 1.$

$\Sigma =$ Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



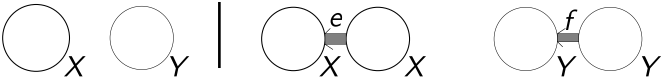
$\partial(e) = 1$

$\partial(f) = 1$

$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.

$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$
$(X \triangleright f) f^{-1} = 1.$

$\Sigma =$ Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise

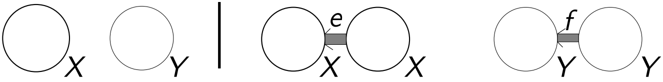


$\partial(e) = 1$
$\partial(f) = 1$

$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.

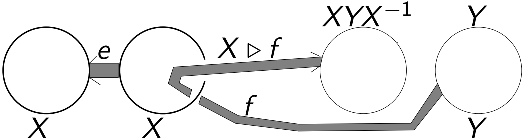
$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$
$(X \triangleright f) f^{-1} = 1.$

Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



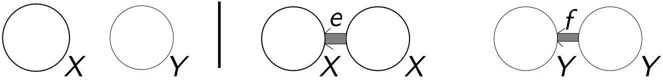
$\partial(e) = 1$
$\partial(f) = 1$

$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.



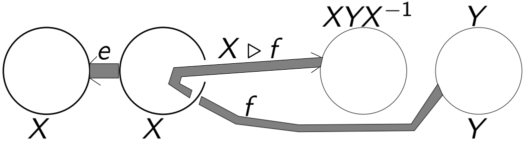
$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$
$(X \triangleright f) f^{-1} = 1.$

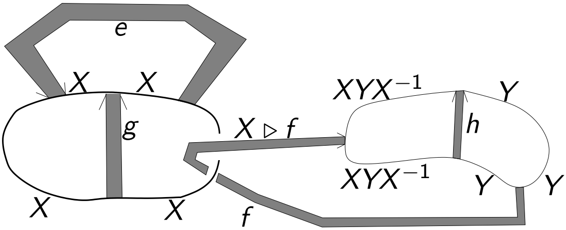
$\Sigma =$ Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



$\partial(e) = 1$
$\partial(f) = 1$

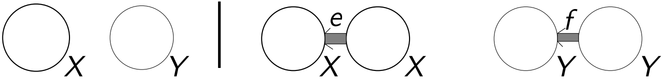
$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.





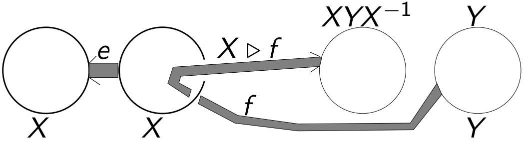
$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$
$(X \triangleright f) f^{-1} = 1.$

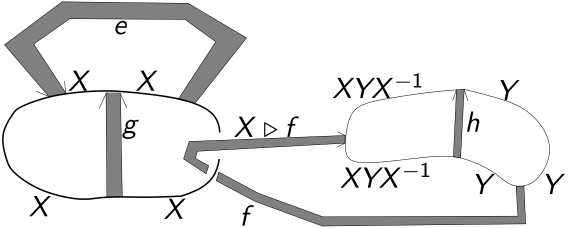
Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



$\partial(e) = 1$
$\partial(f) = 1$

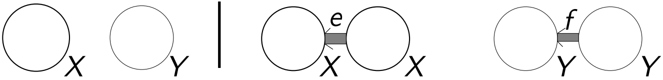
$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.





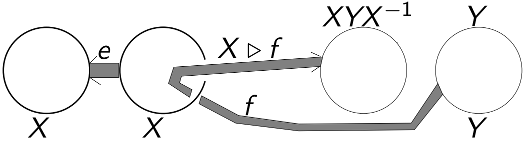
$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\,(X \triangleright f^{-1})\, f = 1$
$(X \triangleright f) f^{-1} = 1.$

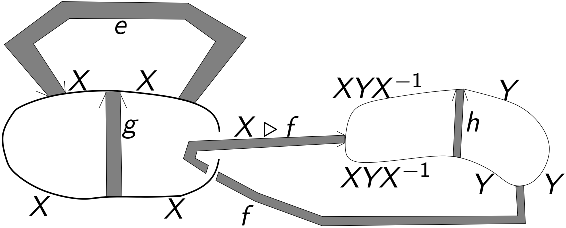
$\Sigma =$ Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



$\partial(e) = 1$
$\partial(f) = 1$

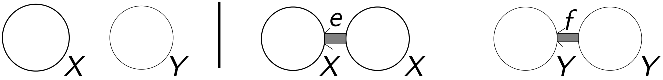
$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.





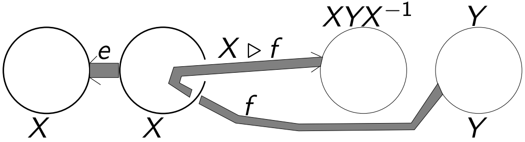
$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\,e^{-1}\,(X \triangleright f^{-1})\,f = 1$
$(X \triangleright f)f^{-1} = 1.$

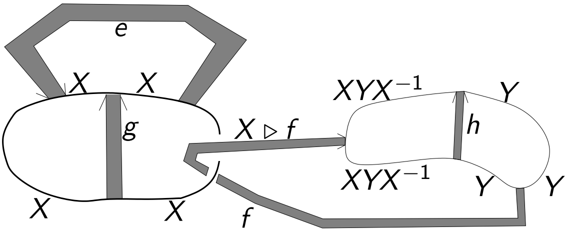
$\Sigma =$ Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



$\partial(e) = 1$
$\partial(f) = 1$

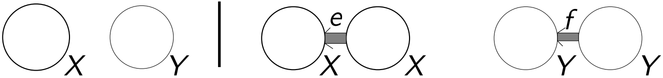
$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.





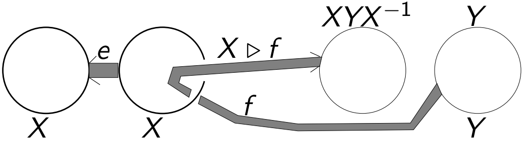
$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\, e^{-1}\, (X \triangleright f^{-1})\ f = 1$
$(X \triangleright f) f^{-1} = 1.$

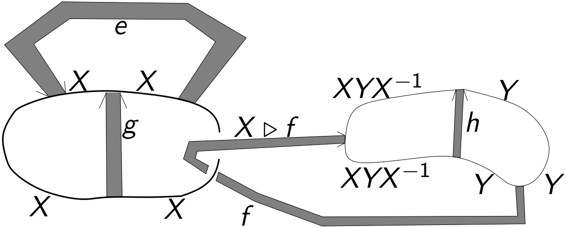
Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



$$\partial(e) = 1$$
$$\partial(f) = 1$$

$X, Y \in \pi_1(M^{(1)})$; $e, f \in \pi_2(M^{(2)}, M^{(1)})$.





$$\partial(g) = 1$$
$$\partial(h) = XYX^{-1}Y^{-1}$$
$$e\, e^{-1}\, (X \triangleright f^{-1})\; f = 1$$
$$(X \triangleright f) f^{-1} = 1.$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto 1\\ h\mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\}/ < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e,f,g,h\} \xrightarrow{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto 1\\ h\mapsto [X,Y]}} \mathcal{F}(\{X,Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1\rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\}/ < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

$\Sigma =$ Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto 1\\ h\mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\} / < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e,f,g,h\} \xrightarrow{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto 1\\ h\mapsto [X,Y]}} \mathcal{F}(\{X,Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1\rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\}/ < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto 1 \\ h \mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\} / < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

$\Sigma=$ Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto 1 \\ h \mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\}/ < f = X.f >$.

Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto 1 \\ h \mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\} / < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e,f,g,h\} \xrightarrow{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto 1\\ h\mapsto [X,Y]}} \mathcal{F}(\{X,Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\}/ < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\Big\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\Big\}(\#E).$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto 1\\ h\mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\}/ < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\big\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\big\}(\#E).$$

Σ= Knotted $T^2 \sqcup T^2$ above. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto 1 \\ h \mapsto [X,Y]}} \mathcal{F}(\{X, Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, g\}/ < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$I_{\mathcal{G}}(M) = \#\Big\{(X, Y, f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\Big\}(\#E)$.

Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:

$\partial(e) = 1$
$\partial(f) = 1$
$\partial(g) = YXY^{-1}X^{-1}$
$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$

Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

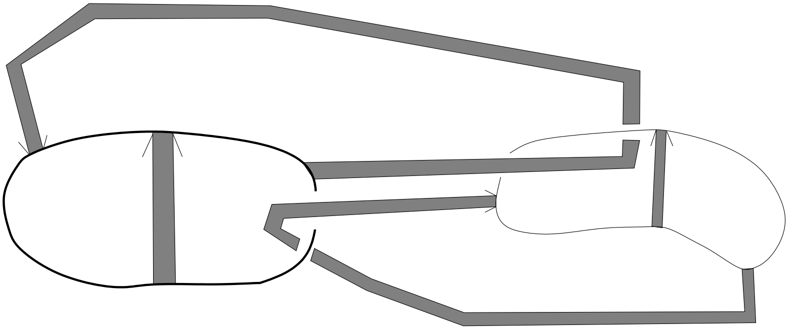
Final stage:

$\partial(e) = 1$
$\partial(f) = 1$
$\partial(g) = YXY^{-1}X^{-1}$
$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$

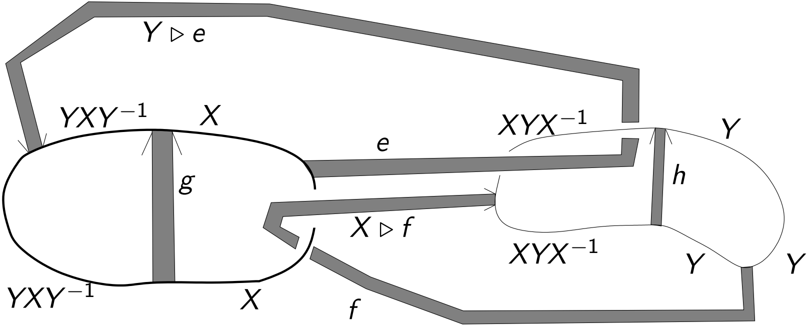
Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:



Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:



$\partial(e) = 1$

$\partial(f) = 1$

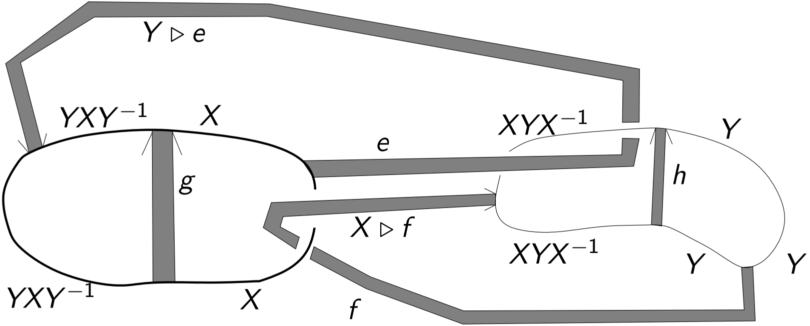
$\partial(g) = YXY^{-1}X^{-1}$

$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$

Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:



$\partial(e) = 1$

$\partial(f) = 1$

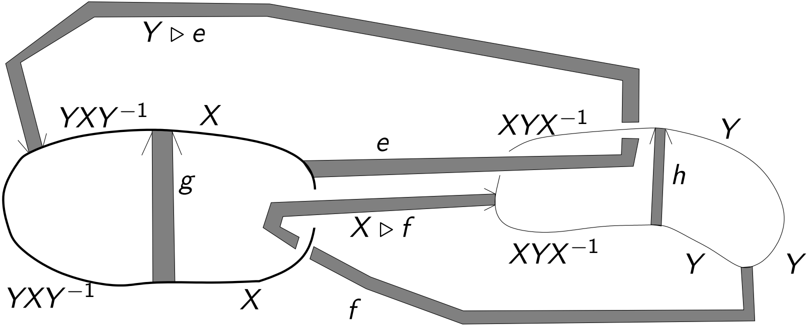
$\partial(g) = YXY^{-1}X^{-1}$

$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$

Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:

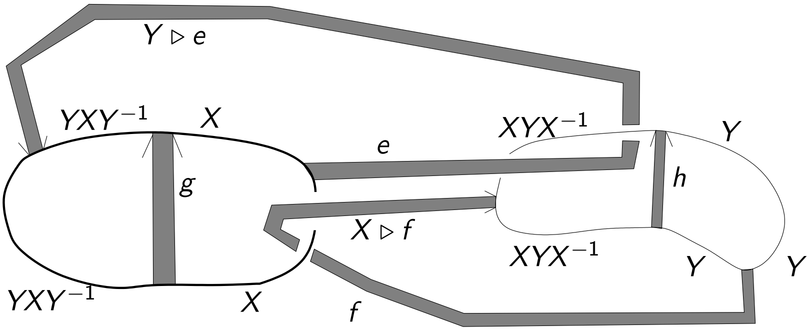


$\partial(e) = 1$
$\partial(f) = 1$
$\partial(g) = YXY^{-1}X^{-1}$
$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$

Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:

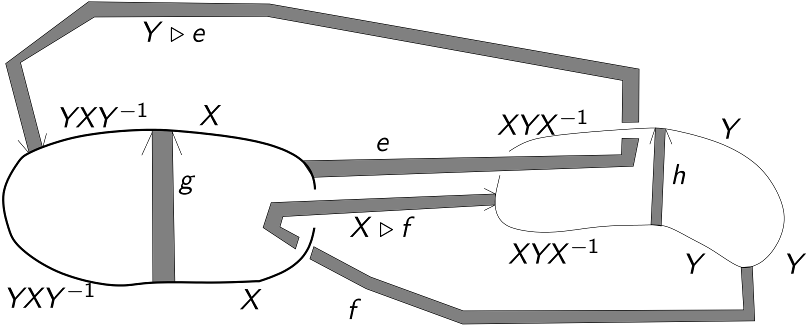


$\partial(e) = 1$
$\partial(f) = 1$
$\partial(g) = YXY^{-1}X^{-1}$
$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$

Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:

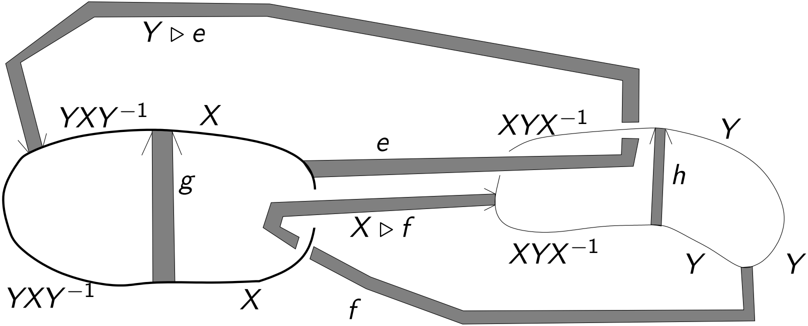


$\partial(e) = 1$
$\partial(f) = 1$
$\partial(g) = YXY^{-1}X^{-1}$
$\partial(h) = XYX^{-1}Y^{-1}$

$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\, f = 1$

Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:



$\partial(e) = 1$
$\partial(f) = 1$
$\partial(g) = YXY^{-1}X^{-1}$
$\partial(h) = XYX^{-1}Y^{-1}$

$$(Y \triangleright e)\, e^{-1}\, (X \triangleright f^{-1})\ f = 1$$

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]\\ h\mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y\triangleright e)\, e^{-1}\\ (X\triangleright f^{-1})\, f\\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1\rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f) + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\left\{(X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY=YX,\\ (Y\triangleright e)-e-(X\triangleright f)+f=0}\right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]\\ h\mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y\triangleright e)\, e^{-1}\\ (X\triangleright f^{-1})\; f\\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1\rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f)\; + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\left\{(X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY=YX,\\ (Y\triangleright e)-e-(X\triangleright f)\;+f=0}\right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e,f,g,h\} \xrightarrow{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]\\ h\mapsto [X,Y]}} \mathcal{F}(X,Y) \;\middle|\; \substack{(Y\triangleright e)\, e^{-1}\\ (X\triangleright f^{-1})\ f\\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f)\ + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \# \left\{ (X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY=YX,\\ (Y\triangleright e)-e-(X\triangleright f)\ +f=0} \right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto [Y,X] \\ h \mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y \triangleright e)\, e^{-1} \\ (X \triangleright f^{-1})\, f \\ = 1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f) + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\left\{ (X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY = YX, \\ (Y \triangleright e) - e - (X \triangleright f) + f = 0} \right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[h \mapsto [X,Y]]{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto [Y,X]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y \triangleright e)\, e^{-1} \\ (X \triangleright f^{-1})\ f \\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f)\ + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\left\{(X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY=YX, \\ (Y \triangleright e) - e - (X \triangleright f)\ + f = 0}\right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' = $ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto [Y,X] \\ h \mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y \triangleright e)\, e^{-1} \\ (X \triangleright f^{-1})\ f \\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f)\ + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \# \left\{ (X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY=YX, \\ (Y \triangleright e) - e - (X \triangleright f)\ + f = 0} \right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U} \left\langle \{e, f, g, h\} \xrightarrow{\substack{e \mapsto 1 \\ f \mapsto 1 \\ g \mapsto [Y,X] \\ h \mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y \triangleright e)\, e^{-1} \\ (X \triangleright f^{-1})\; f \\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f)\; + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \# \left\{ (X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY = YX, \\ (Y \triangleright e) - e - (X \triangleright f)\; + f = 0} \right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e,f,g,h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]\\ h\mapsto [X,Y]}} \mathcal{F}(X,Y) \;\middle|\; \substack{(Y\triangleright e)\, e^{-1}\\ (X\triangleright f^{-1})\ f\\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f) \ + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\left\{(X, Y, e, f) \in G^2 \times E^2 \mid {}_{(Y\triangleright e)-e-(X\triangleright f)\ +f=0}^{XY=YX,}\right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' =$ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]\\ h\mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y\triangleright e)\, e^{-1}\\ (X\triangleright f^{-1})\ f\\ =1} \right\rangle$$

$\pi_1(M) = \langle \{X, Y\} | [X, Y] = 1 \rangle$, free abelian group on X and Y.

$$\pi_2(M) = \frac{\mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e, f, m\}}{< (Y \triangleright e) - e - (X \triangleright f) \ + f = 0 >}.$$

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\left\{(X, Y, e, f) \in G^2 \times E^2 \mid \substack{XY=YX,\\ (Y\triangleright e)-e-(X\triangleright f)\ +f=0}\right\}.$$

$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

More results on $I_{\mathcal{G}}(S^4 \setminus \Sigma)$

Let $\mathcal{G} = (\partial\colon E \to G)$ be a finite crossed module.

Recall $I_{\mathcal{G}}(M) = \frac{1}{\# E^{b_1(M^1)}} \# \operatorname{hom}(\Pi_2(M, M^1), \mathcal{G})$

- The invariant of knotted surfaces:

$$\Sigma \mapsto I_{\mathcal{G}}(S^4 \setminus \Sigma)$$

is able to separate between pairs of knotted surfaces with different knot groups. (Varying $\mathcal{G}$.)

More results on $I_{\mathcal{G}}(S^4 \setminus \Sigma)$

Let $\mathcal{G} = (\partial\colon E \to G)$ be a finite crossed module.

Recall $I_{\mathcal{G}}(M) = \frac{1}{\# E^{b_1(M^1)}} \# \hom(\Pi_2(M, M^1), \mathcal{G})$

- The invariant of knotted surfaces:

$$\Sigma \mapsto I_{\mathcal{G}}(S^4 \setminus \Sigma)$$

is able to separate between pairs of knotted surfaces with different knot groups. (Varying $\mathcal{G}$.)

More results on $I_{\mathcal{G}}(S^4 \setminus \Sigma)$

Let $\mathcal{G} = (\partial\colon E \to G)$ be a finite crossed module.

Recall $I_{\mathcal{G}}(M) = \frac{1}{\# E^{b_1(M^1)}} \# \hom(\Pi_2(M, M^1), \mathcal{G})$

- The invariant of knotted surfaces:

$$\Sigma \mapsto I_{\mathcal{G}}(S^4 \setminus \Sigma)$$

  is able to separate between pairs of knotted surfaces with different knot groups. (Varying $\mathcal{G}$.)

More results on $I_{\mathcal{G}}(S^4 \setminus \Sigma)$

Let $\mathcal{G} = (\partial\colon E \to G)$ be a finite crossed module.

Recall $I_{\mathcal{G}}(M) = \frac{1}{\# E^{b_1(M^1)}} \# \hom(\Pi_2(M, M^1), \mathcal{G})$

- The invariant of knotted surfaces:

$$\Sigma \mapsto I_{\mathcal{G}}(S^4 \setminus \Sigma)$$

  is able to separate between pairs of knotted surfaces with different knot groups. (Varying $\mathcal{G}$.)

More results on $I_{\mathcal{G}}(S^4 \setminus \Sigma)$

Let $\mathcal{G} = (\partial\colon E \to G)$ be a finite crossed module.

Recall $I_{\mathcal{G}}(M) = \frac{1}{\# E^{b_1(M^1)}} \# \hom(\Pi_2(M, M^1), \mathcal{G})$

- The invariant of knotted surfaces:

$$\Sigma \mapsto I_{\mathcal{G}}(S^4 \setminus \Sigma)$$

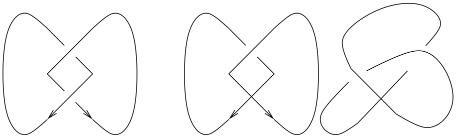
  is able to separate between pairs of knotted surfaces with different knot groups. (Varying $\mathcal{G}$.)

Welded knots

Recall that Welded knots are virtual knot diagrams like:

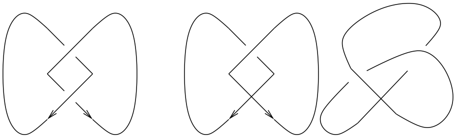
Welded knots

Recall that Welded knots are virtual knot diagrams like:



Welded knots

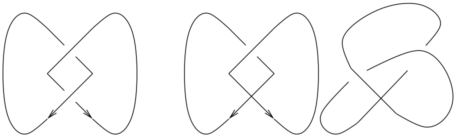
Recall that Welded knots are virtual knot diagrams like:



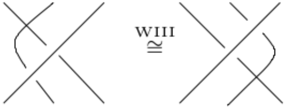
Modulo relations:

Welded knots

Recall that Welded knots are virtual knot diagrams like:

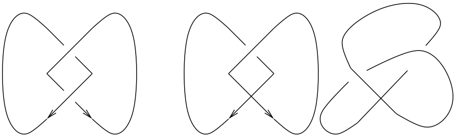


Modulo relations:

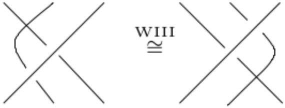


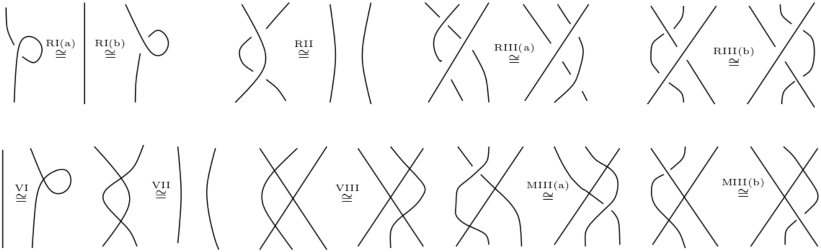
Welded knots

Recall that Welded knots are virtual knot diagrams like:



Modulo relations:





The tube map

- Recall Shin Satoh's "tube-map"

$$Tube \colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications 9 (2000), 531-542.

The tube map

- Recall Shin Satoh's "tube-map"

$$\mathit{Tube}\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications 9 (2000), 531-542.

The tube map

- Recall Shin Satoh's "tube-map"

$$\mathit{Tube} \colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications 9 (2000), 531-542.

The tube map

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications 9 (2000), 531-542.

The tube map

- Recall Shin Satoh's "tube-map"

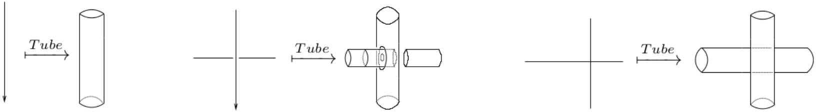
$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications 9 (2000), 531-542.

The tube map

- Recall Shin Satoh's "tube-map"

$$\mathit{Tube}\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

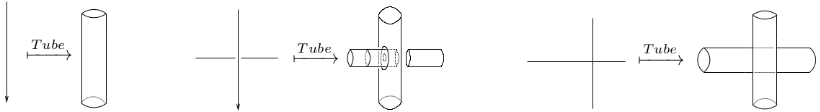


S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications 9 (2000), 531-542.

The tube map

- Recall Shin Satoh's "tube-map"

$$\textit{Tube}\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$



S. Satoh, Virtual knot presentation of ribbon torus-knots, J. Knot Theory Ramifications 9 (2000), 531-542.

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$. The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

.

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.

The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

.

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.

Proof essentially in:

JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

.

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$. The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

.

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:

.

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

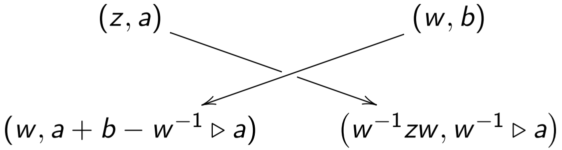
- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:



.

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

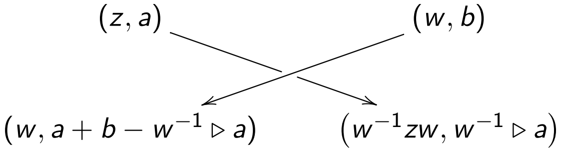
- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:



.

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

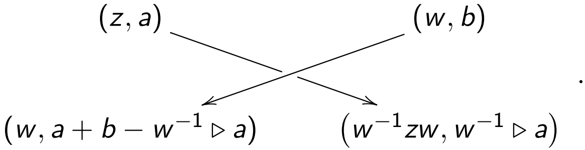
- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:



So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

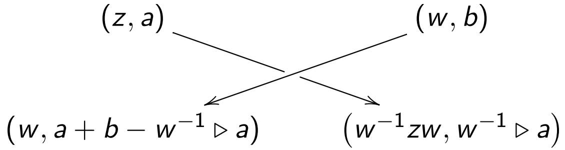
- Recall Shin Satoh's "tube-map"

$$Tube\colon \{\text{Welded links}\} \to \{\text{Knotted Tori in } S^4\}$$

Theorem: Suppose $\mathcal{G} = (A \to G, \triangleright)$ is finite and $\partial(A) = \{1_G\}$.
The welded knot invariant

$$K \mapsto I_{\mathcal{G}}(S^4 \setminus Tube(K))$$

is computed from a biquandle with underlying set $G \times A$:



.

So A is an abelian G-module, $z, w \in G$, $a, b \in A$.
Proof essentially in:
JFM., Kauffman L.H.: Invariants of Welded Virtual Knots Via Crossed Module Invariants of Knotted Surfaces, Comp. Math. 2008

Biquandles and $I_{\mathcal{G}}$

The biquandle below (A an abelian G-module):

.

has since been revisited in:

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. ATMP 2019.

Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

- Damiani C, JFM , Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898.

A higher order version of Artin representation defined.

Biquandles and $I_{\mathcal{G}}$

The biquandle below (A an abelian G-module):

.

has since been revisited in:

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. ATMP 2019.

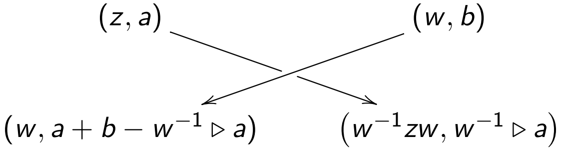
Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

- Damiani C, JFM , Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898.

A higher order version of Artin representation defined.

Biquandles and $I_{\mathcal{G}}$

The biquandle below (A an abelian G-module):



.

has since been revisited in:

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. ATMP 2019.

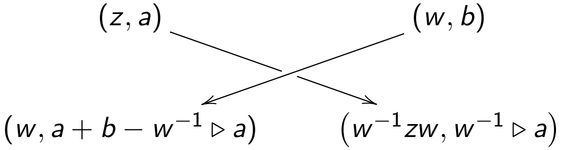
Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

- Damiani C, JFM , Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898.

A higher order version of Artin representation defined.

Biquandles and $I_{\mathcal{G}}$

The biquandle below (A an abelian G-module):



.

has since been revisited in:

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. ATMP 2019.

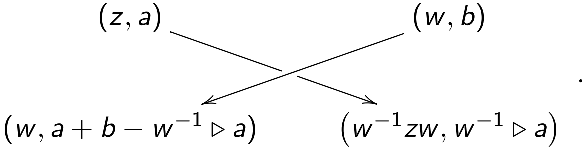
Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

- Damiani C, JFM , Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898.

A higher order version of Artin representation defined.

Biquandles and $I_{\mathcal{G}}$

The biquandle below (A an abelian G-module):



has since been revisited in:

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. ATMP 2019.

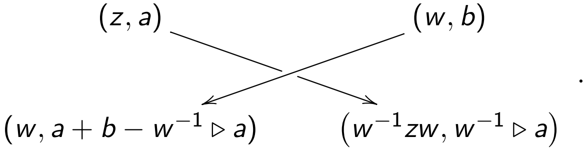
Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

- Damiani C, JFM , Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898.

A higher order version of Artin representation defined.

Biquandles and $I_{\mathcal{G}}$

The biquandle below (A an abelian G-module):



has since been revisited in:

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. ATMP 2019.

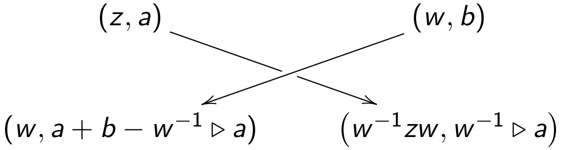
Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

- Damiani C, JFM , Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898.

A higher order version of Artin representation defined.

Biquandles and $I_{\mathcal{G}}$

The biquandle below (A an abelian G-module):



.

has since been revisited in:

- Bullivant A, Martin P, and JFM: Representations of the Loop Braid Group and Aharonov-Bohm like effects in discrete (3+1)-dimensional higher gauge theory. ATMP 2019.

Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

- Damiani C, JFM , Martin P: On a canonical lift of Artin's representation to loop braid groups. arXiv:1912.11898.

A higher order version of Artin representation defined.

THANKS!