Crossed modules, homotopy 2-types, knotted surfaces and welded knots

Topology Seminar (Lille).

2nd April 2021

João Faria Martins (University of Leeds)

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Thanks due to Tim Porter, Paul Martin, Alex Bullivant, Celeste Damiani, Gustavo Granja, Louis Kauffman.

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Knot complements are aspherical!

Let K be a (one-component) piecewise linear / smooth knot in S^3

- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.

- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g.

Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\text{-}\mathbf{types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

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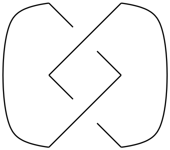

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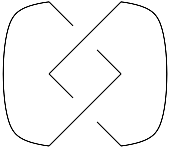

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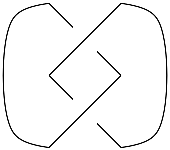

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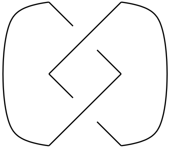

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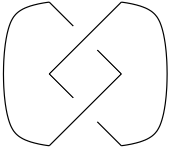

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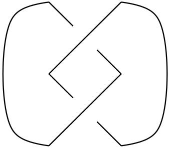

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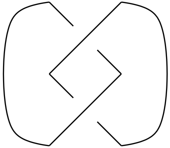

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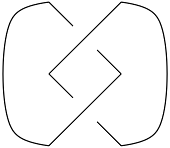
- Papakyriakopoulos theorem: $S^3 \setminus K$ is an aspherical space.
- *Asphericity* means that: $\pi_i(S^3 \setminus K) = 0$, if $i \geq 2$.
- More generally $S^3 \setminus L$ is aspherical if $L \subset S^3$ is a *non-splittable* link.

E.g. L =



Definition: (n-type) Let $n \in \mathbb{Z}_0^+$.

An *n-type* is a path-connected pointed space $X = (X, *)$ such that:

1. X is homeomorphic to a CW-complex, with $*$ being a 0-cell. (Frequenly omitted in model categories literature.)
2. $\pi_i(X) = 0$, if $i > n$.

Let $\{n\textbf{-types}\}$ be the category with objects the n-types.

Given two n-types X and Y,

morphisms $X \to Y$ are pointed homotopy classes of pointed maps.

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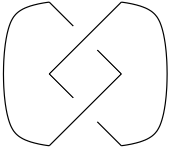

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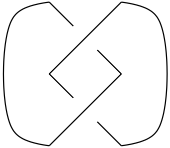

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1-types and knot complements

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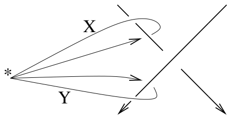

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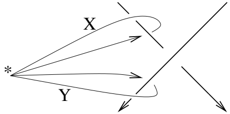

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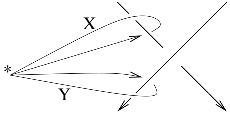

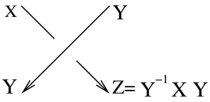

Beyond 1-types: complements of knotted surfaces $\Sigma \subset S^4$

Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
(Any genus, any number of components, possibly non-orientable.)

Fact: $S^4 \setminus \Sigma$ need not be aspherical.

Also $\pi_1(S^4 \setminus \Sigma)$ does not classify $S^4 \setminus \Sigma$ up to homotopy.

We need to look at 'higher order' homotopy type information in order to classify $S^4 \setminus \Sigma$ up to homotopy.

Let us look at *the homotopy 2-type* $\mathcal{P}_2(S^4 \setminus \Sigma)$ of $S^4 \setminus \Sigma$.

This topological space $\mathcal{P}_2(S^4 \setminus \Sigma)$ is obtained from $S^4 \setminus \Sigma$ by functorially killing all homotopy groups π_i, for $i \geq 3$.

I.e. we throw away homotopy theoretical information of order ≥ 3.

Hence $\mathcal{P}_2(S^4 \setminus \Sigma)$ is a 2-type.

Theorem

Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

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Category of 2-types is equivalent to homotopy category of 2-groups.

... *To be explained later.*

We will see 2-groups as being represented by crossed modules.

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Let $\Sigma \subset S^4$ be a closed surface smoothly embedded in S^4.
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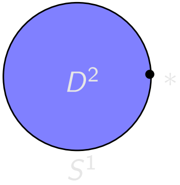

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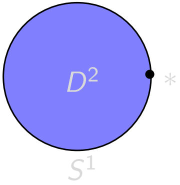

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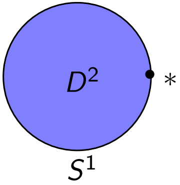

Free crossed modules

Let V be a set, G a group. Consider a set map $\partial_0 \colon V \to G$.
We can define the "free crossed module on ∂_0", denoted

$$\mathcal{U}\langle \partial_0 \colon V \to G \rangle = (\partial \colon \mathcal{F}(V \xrightarrow{\partial_0} G) \longrightarrow G, \triangleright).$$

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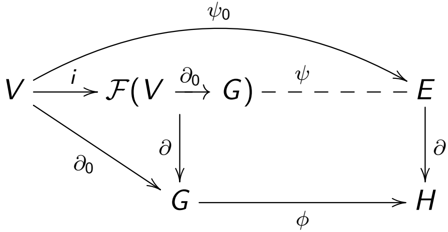

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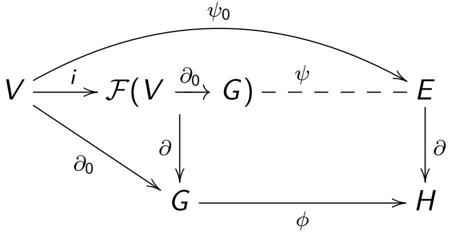

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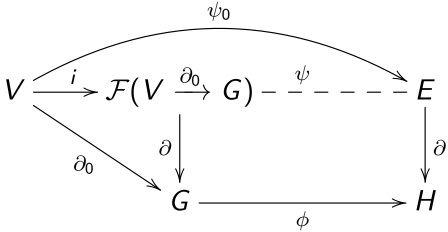

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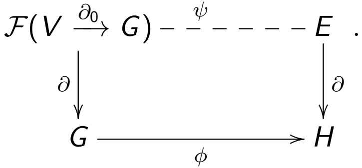

Free crossed modules and Whitehead theorem

$$Y = X \cup c_1 \cup c_2.$$

Whitehead theorem: If Y is obtained from X by attaching 2-cells, then $\Pi_2(Y, X)$ is free on the attaching maps $\{2-cells\} \xrightarrow{\partial_0} \pi_1(X)$.

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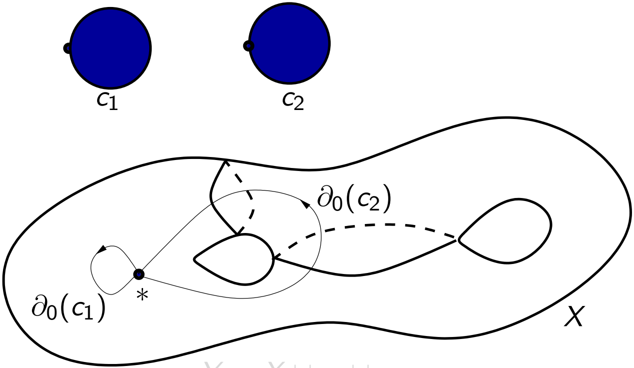

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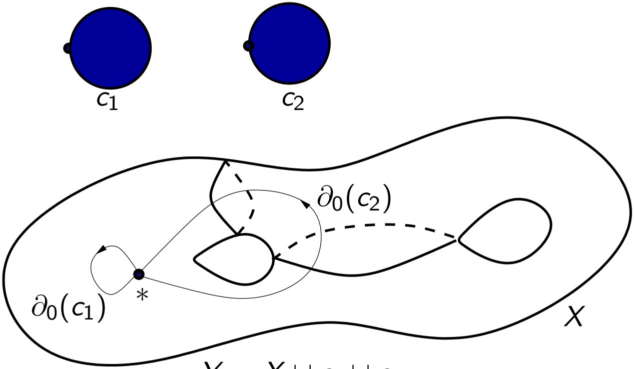

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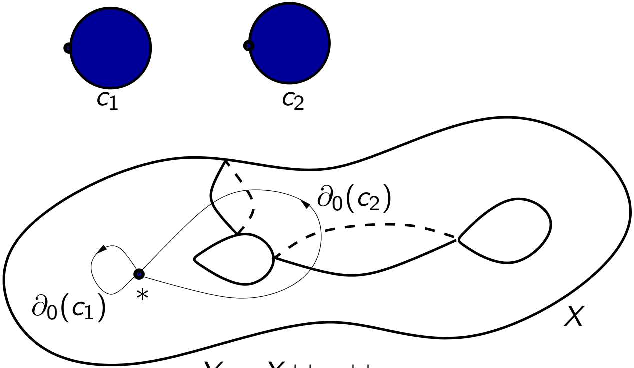

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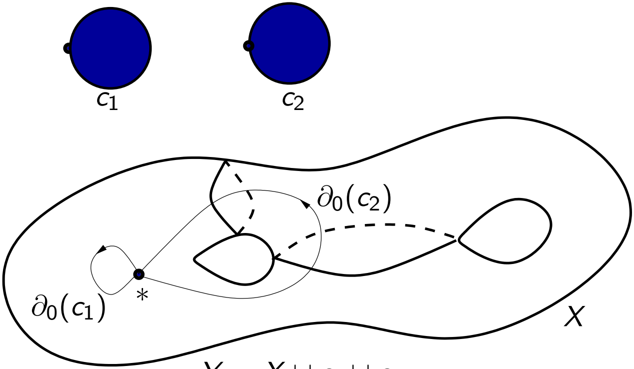

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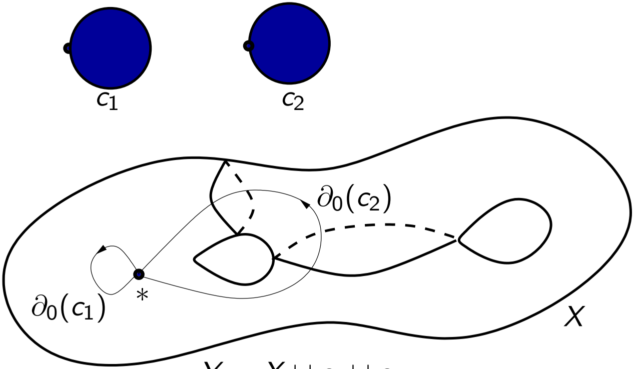

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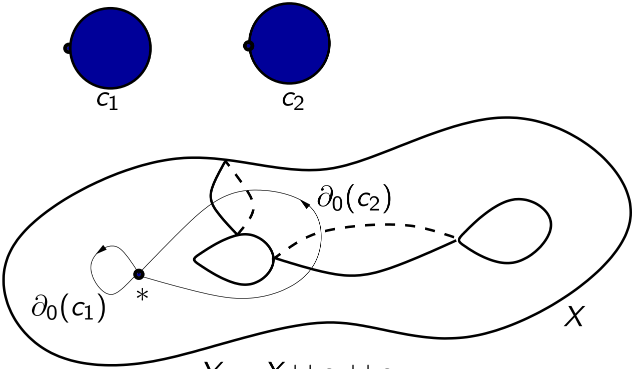

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A crossed module $\mathcal{G} = (E \xrightarrow{\partial} G)$ contains a short complex $E \to G$.

Given $\mathcal{G}$ and $\mathcal{G}' = (E' \to G')$, $\exists$ notion of homotopy of maps $\mathcal{G} \to \mathcal{G}'$.

Homotopies are built on group derivations $s \colon G \to E'$.

Fact: We have category $\{$**Cof-Crossed Modules**$\}/\simeq$.
Objects are crossed modules $\mathcal{G} = (\partial \colon E \to F)$; F a free group.
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Ho($\{$**Crossed Modules**$\}$*) is equivalent to* $\{$**2-types**$\}$.

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I.e.:
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Theorem (Whitehead / MacLane 1950 PNAS)

1. *When restricted to 2-types, Π_2 is an equivalence of categories.*
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by generators and relations. (In the world of crossed modules.)

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Presentation of $\Pi_2(X, X^1)$ by generators and relations

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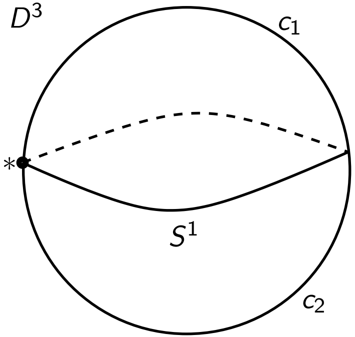

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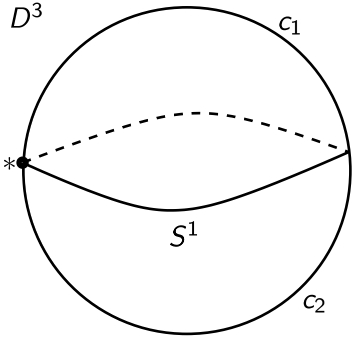

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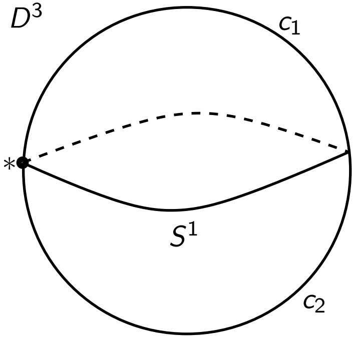

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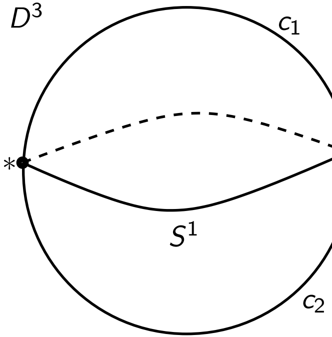

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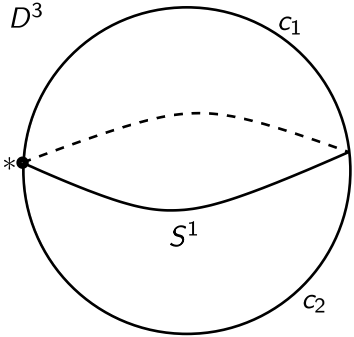

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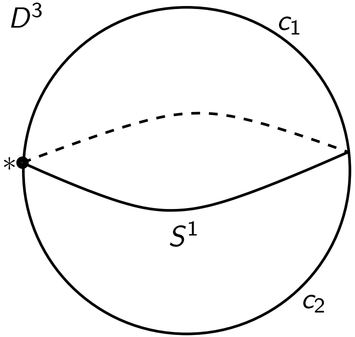

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The homotopy invariant $I_{\mathcal{G}}$.

Up to homotopy $\Pi_2(X, X^1)$ doesn't depend on CW-decomposition of X
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Proposition Let $\mathcal{G} = (\partial \colon E \to G, \triangleright)$ be a finite crossed module.
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Calculation of $\Pi_2(S^4 \setminus \Sigma)$, Σ a knotted surface

Let $\Sigma \subset S^4 = \mathbb{R}^4 \cup \{\infty\}$ be a knotted surface.
(Any genus, any number of components.)
Suppose the projection on the t-variable is a Morse function in Σ.
To simplify, suppose critical points appear in increasing order.
Let $\Sigma_t = \Sigma \cap (\mathbb{R}^3 \times \{t\})$, called the "still of Σ at t".

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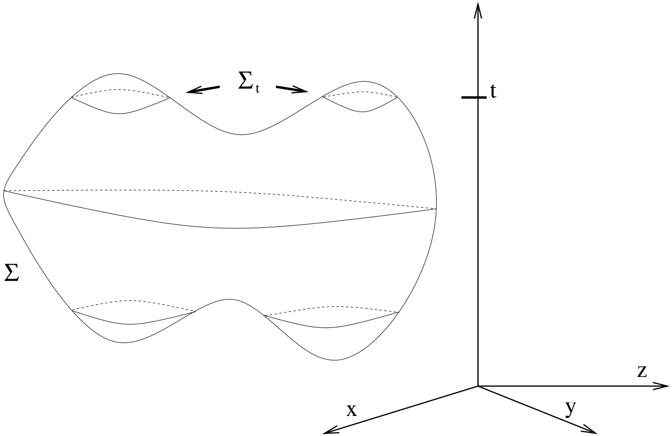

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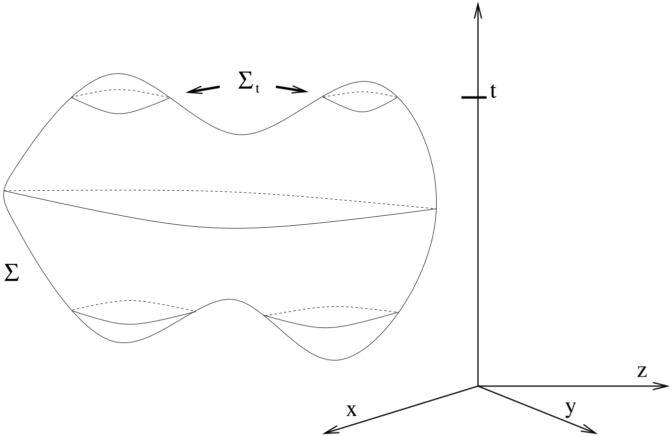

Handle decomposition (fat CW-decomposition) of $M = S^4 \setminus \Sigma$

Let $M^{(i)}$ be union of handles of index $\leq i$.

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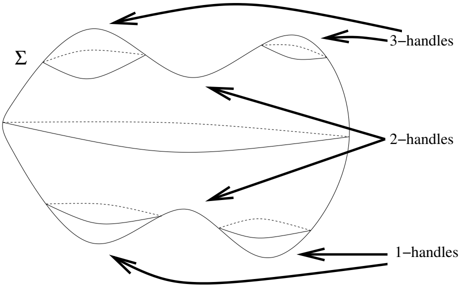

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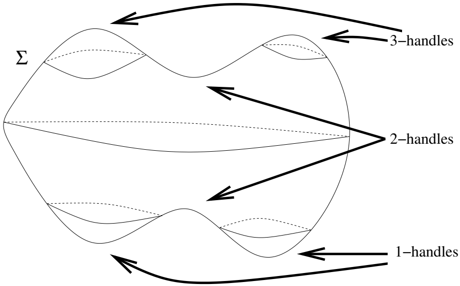

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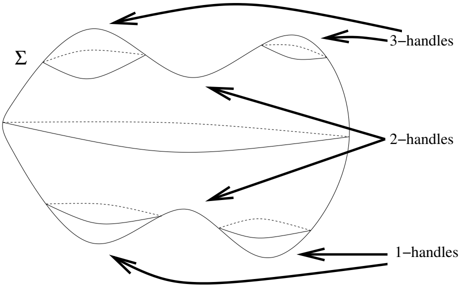

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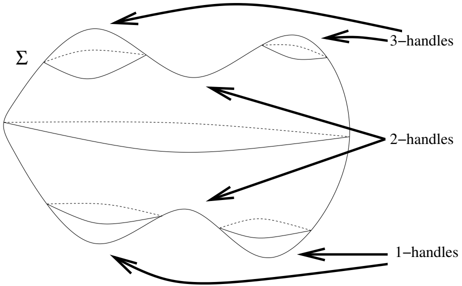

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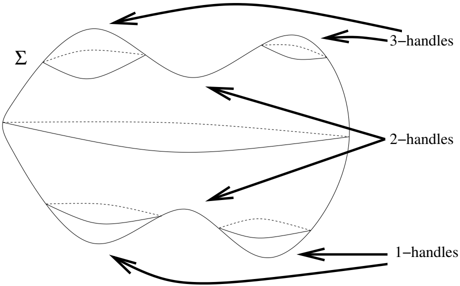

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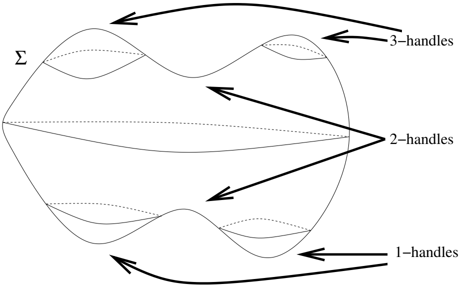

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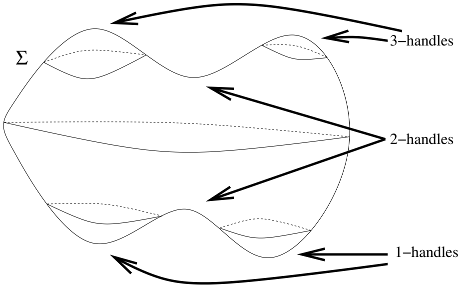

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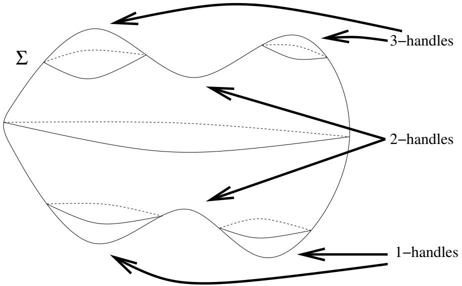

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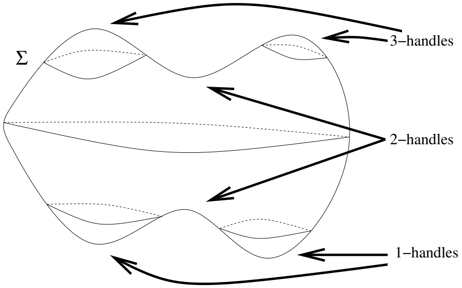

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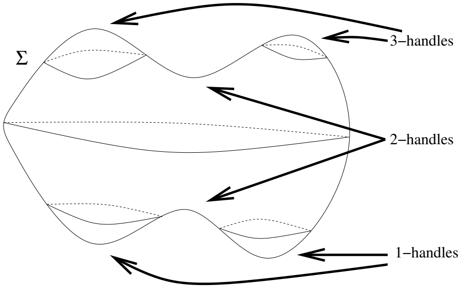

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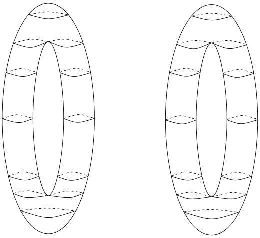

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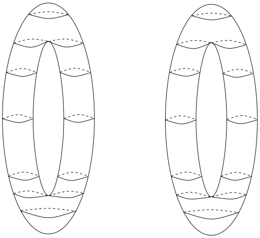

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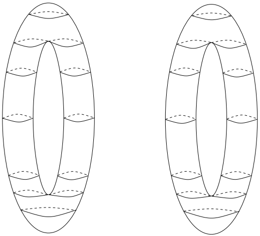

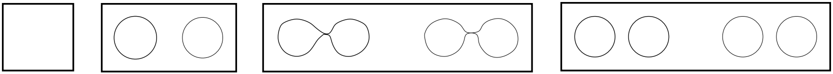

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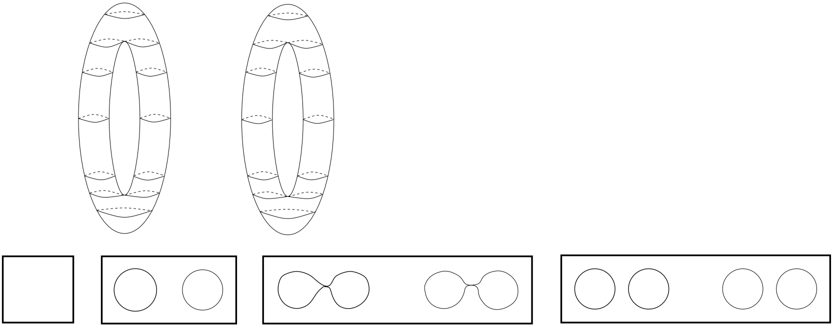

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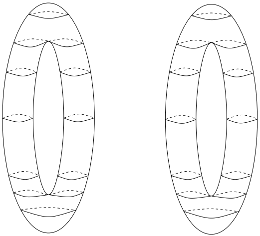

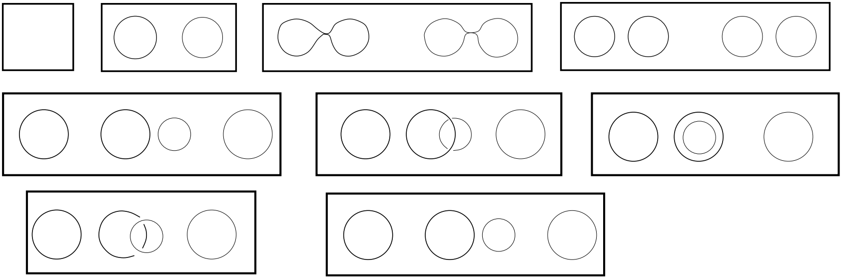

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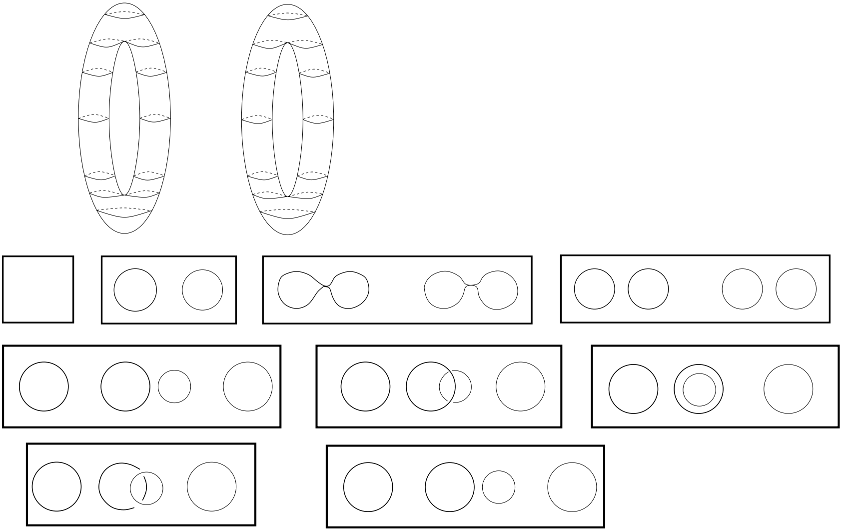

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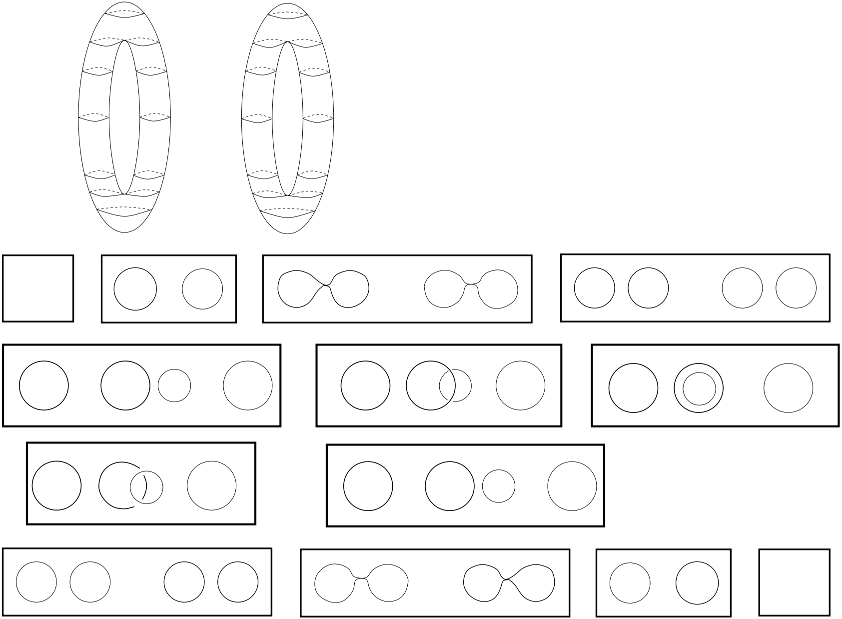

Free generators of $\pi_1(M^{(1)})$ at minimal points

Let $\Sigma \subset S^4$, oriented surface, Morse conditions as above.
Let $M = S^4 \setminus \Sigma$. Let $M^{(i)}$ be union of handles of degree $\leq i$.

Locally, an oriented minimal point looks like:

A minimal point yields a 1-handle of M.
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it:

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as:

As the movie evolves, throughout an isotopy, we colour the link arcs of each still Σ_t by the generators of $\pi_1(M^{(1)})$ they represent.
There are relations between generators at different times. For $R2$:

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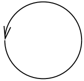

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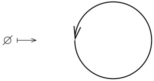

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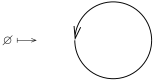

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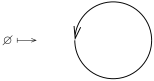

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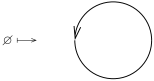

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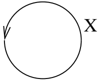

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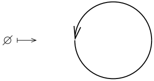

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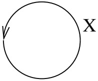

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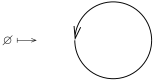

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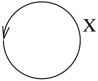

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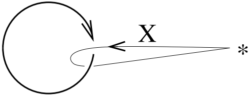

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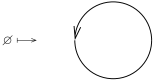

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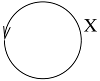

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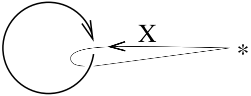

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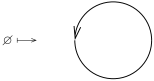
Locally, an oriented minimal point looks like: 

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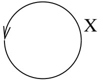
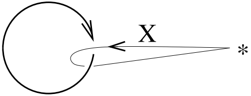
Hence a free generator of $X \in \pi_1(M^{(1)})$. Denote it: 

Concretely, $X \in \pi_1(M^{(1)})$ can be defined as: 

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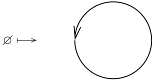

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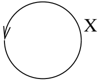

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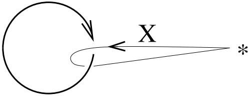

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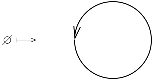

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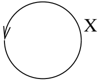

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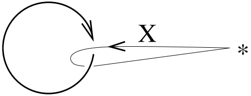

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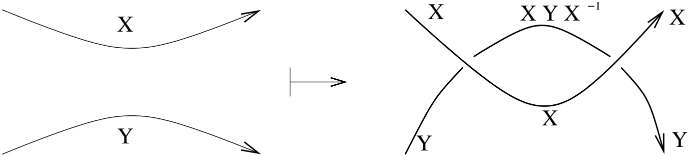

Free generators of $\Pi_2(M^{(2)}, M^{(1)})$ at saddle points

Locally, an (oriented) saddle point looks like:

When passing saddle point, add a 'band', kept throughout movie:
This band will later bookkeep where the saddle point was made, and the attaching region of corresponding 2-handle of M.

Each band gives free crossed module generator $e \in \pi_2(M^{(2)}, M^{(1)})$.

Bands are to be kept and evolve throughout the rest of the movie.
Each arc of a band in a projection gives element of $\pi_2(M^{(2)}, M^{(1)})$.

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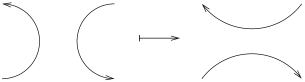

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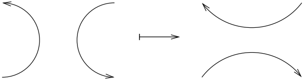

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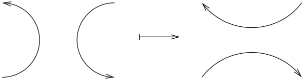

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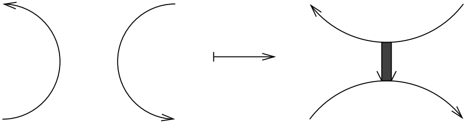

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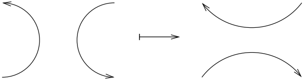

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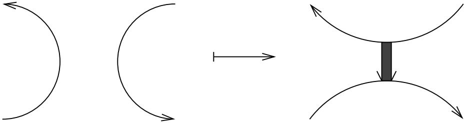

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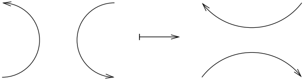

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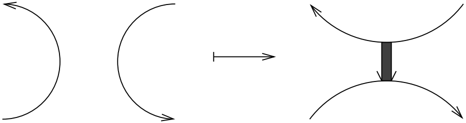

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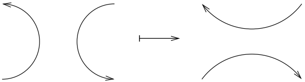

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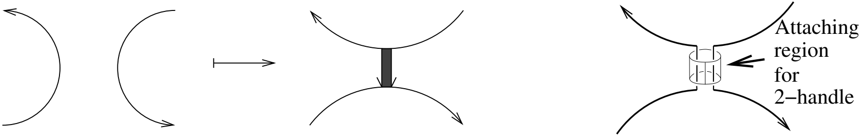

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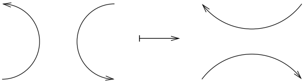

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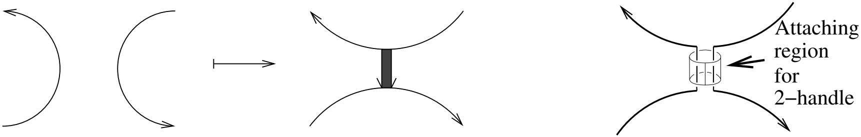

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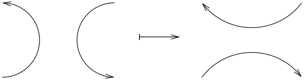

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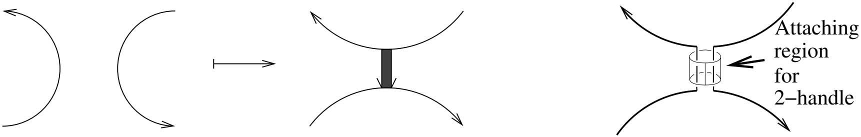

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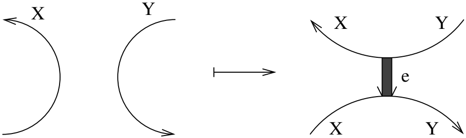

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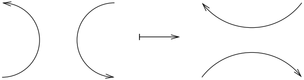

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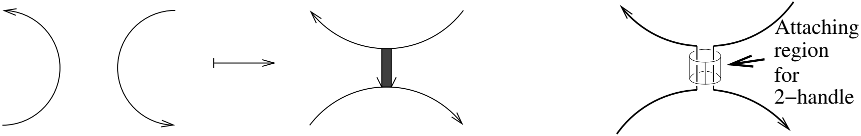

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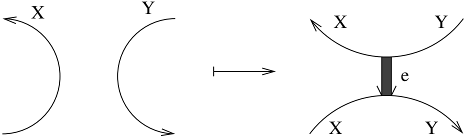

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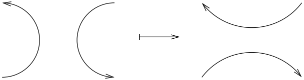

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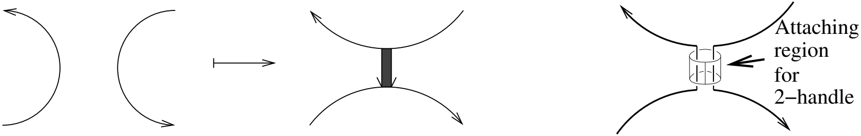

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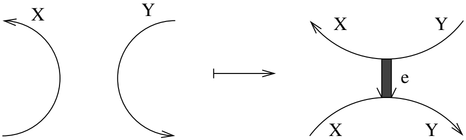

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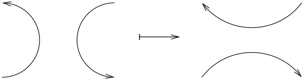

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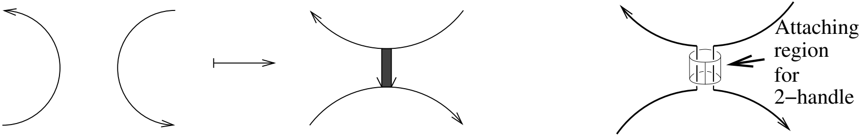

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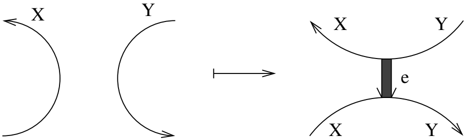

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Some bands will possibly be present.
Before maximal point, configuration looks like:

In this case the 2-relations are as below:

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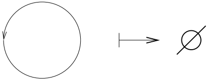
Locally, an oriented maximal point looks like: 

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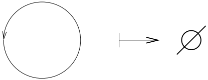
Locally, an oriented maximal point looks like: 

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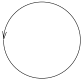
Locally, an oriented maximal point looks like:  $\longmapsto \emptyset$

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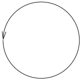
Locally, an oriented maximal point looks like:  $\mapsto \emptyset$

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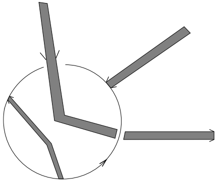
Before maximal point, configuration looks like: 

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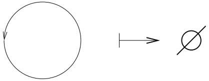
Locally, an oriented maximal point looks like: 

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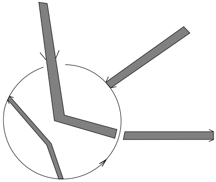
Before maximal point, configuration looks like: 

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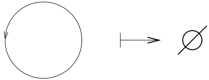

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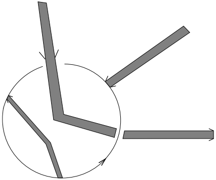

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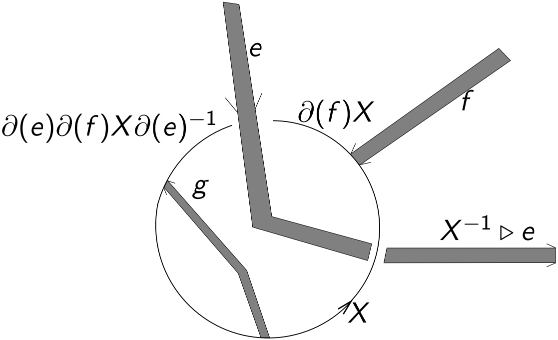

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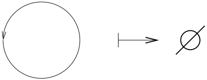

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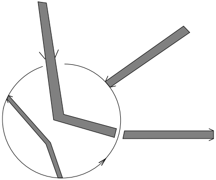

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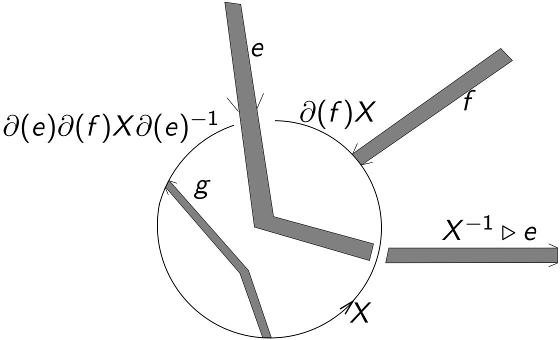

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A movie for a knotted union Σ of two tori

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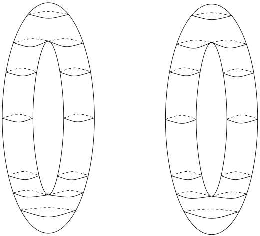

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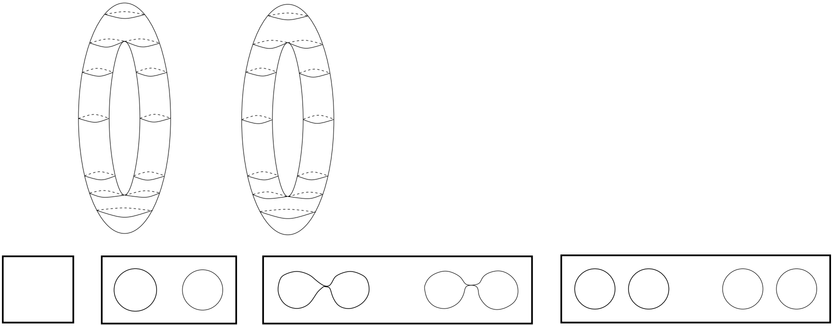

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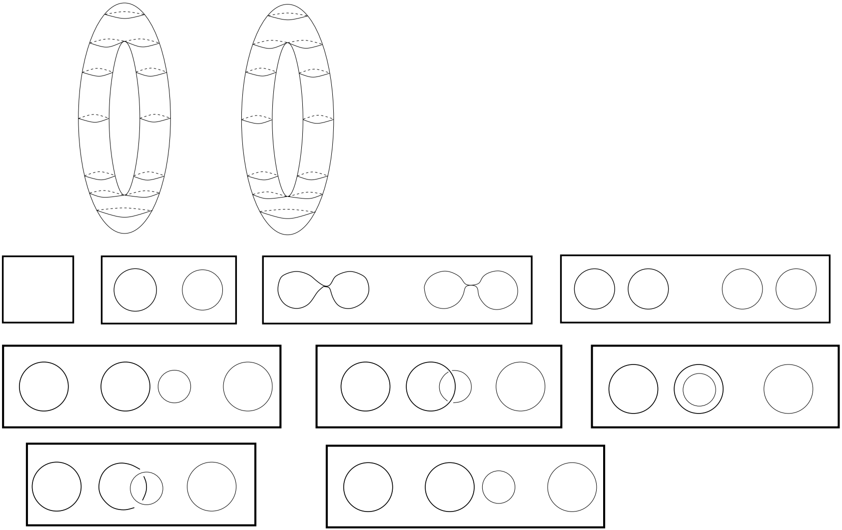

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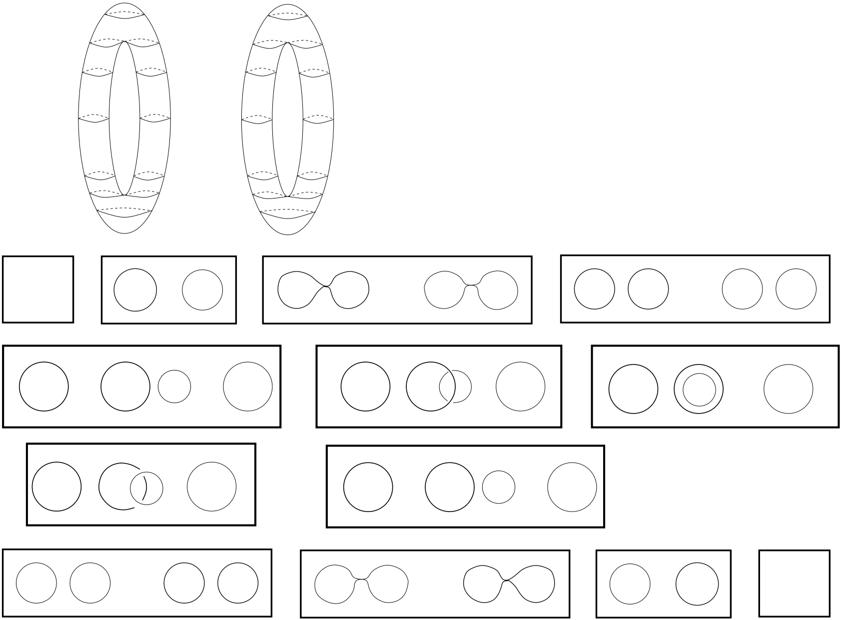

Σ = Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise

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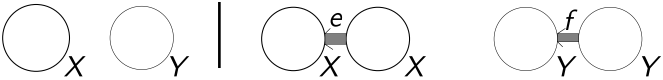

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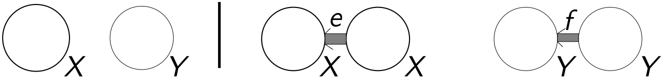

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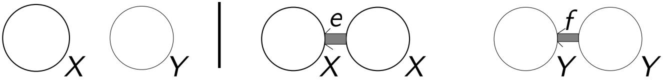

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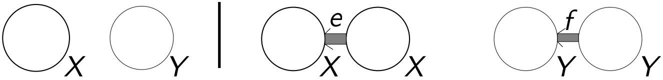

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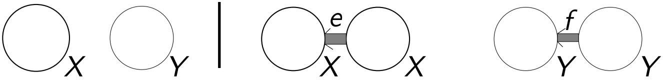

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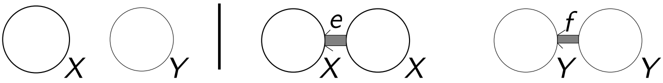

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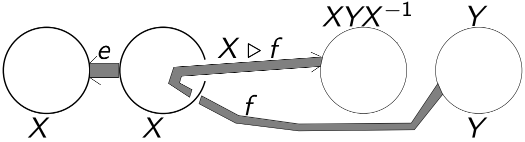

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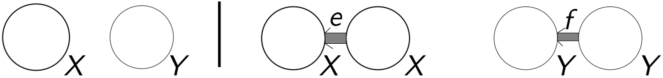

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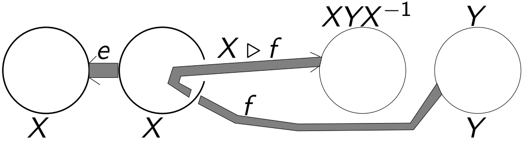

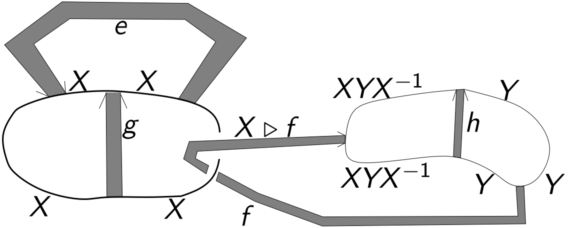

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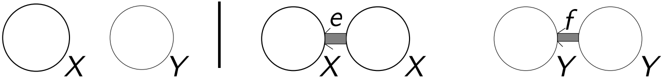

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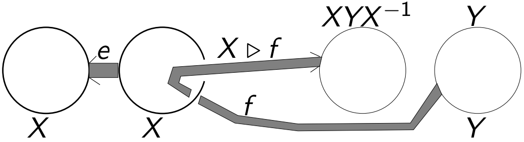

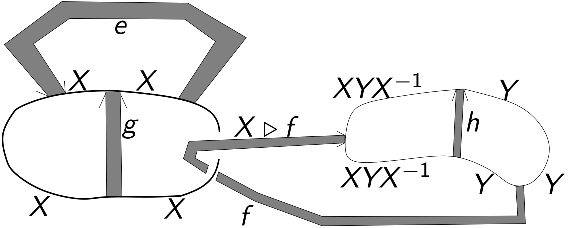

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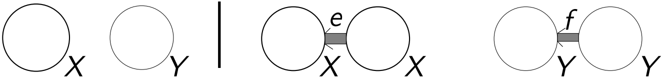

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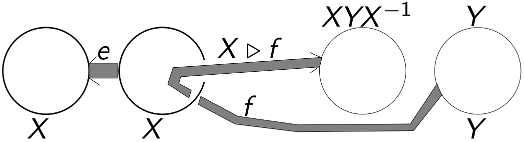
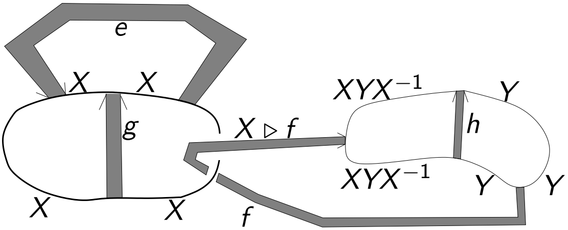
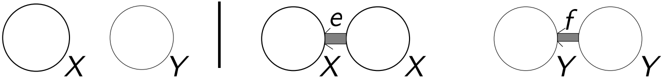




$\partial(g) = 1$
$\partial(h) = XYX^{-1}Y^{-1}$
$e\,e^{-1}\,(X \triangleright f^{-1})\,f = 1$
$(X \triangleright f)f^{-1} = 1.$

$\Sigma =$ Knotted $T^2 \sqcup T^2$ above. Circles oriented counterclockwise



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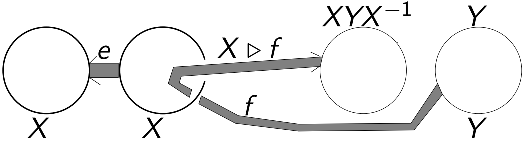

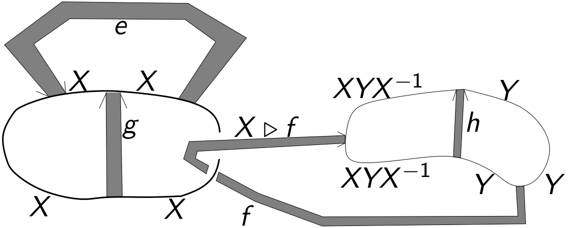

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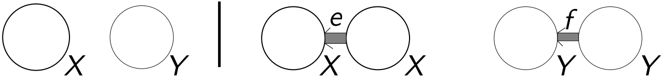

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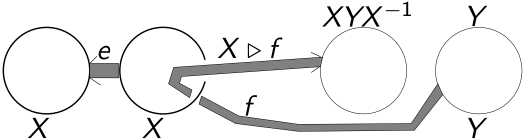

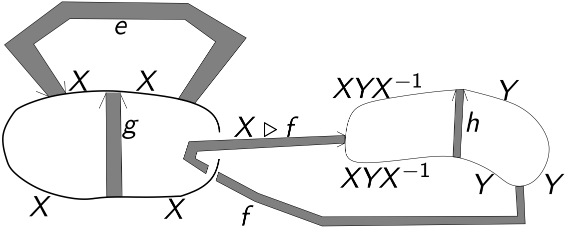

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Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e,f,g,h\} \xrightarrow{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto 1\\ h\mapsto [X,Y]}} \mathcal{F}(\{X,Y\}) \mid f = X \triangleright f \right\rangle$$

$\pi_1(M) = \langle \{X,Y\} | [X,Y] = 1 \rangle$, free abelian group on X and Y.

$\pi_2(M) = \mathbb{Z}[X, X^{-1}, Y, Y^{-1}]\{e,f,g\} / < f = X.f >$.
Quotient of the free module over the algebra of Laurent polynomials in X and Y, on the generators e, f, g, by the relation $f = X.f$.

If $\mathcal{G} = (E \to G, \triangleright)$ is finite and $\partial(E) = \{1_G\}$ then:

$$I_{\mathcal{G}}(M) = \#\{(X,Y,f) \in G \times G \times E \mid XY = YX, f = X \triangleright f\}(\#E).$$

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Another example $\Sigma' =$ Spun Hopf Link, a knotted $T^2 \sqcup T^2$

Final stage:

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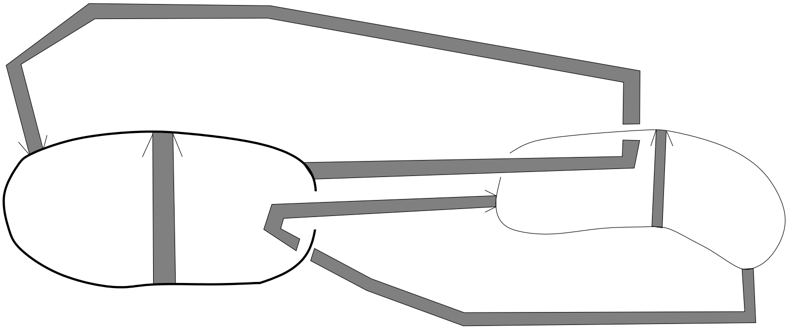

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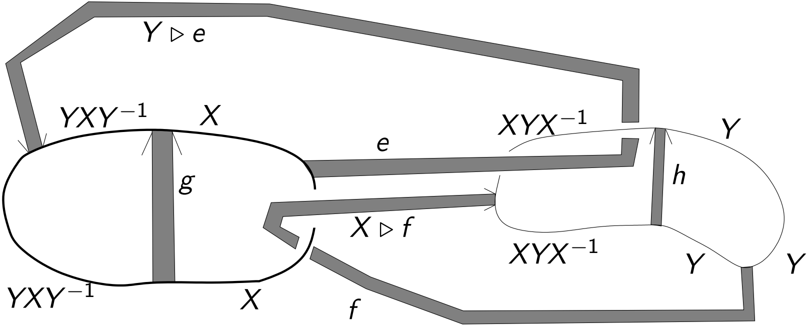

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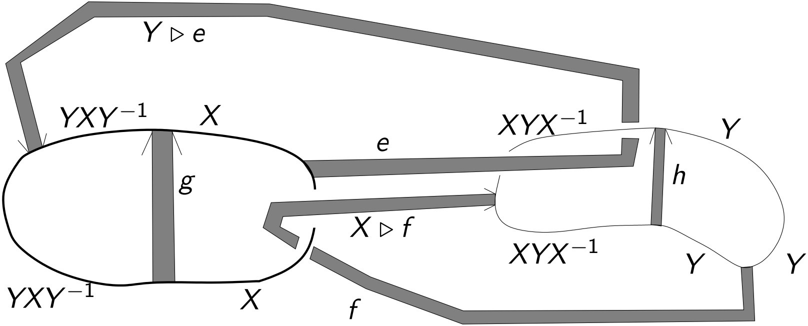

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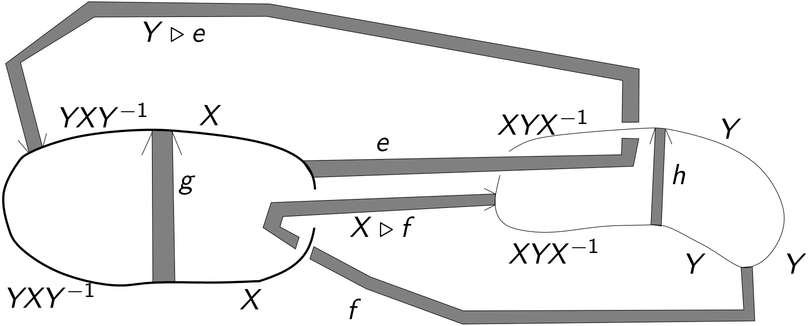

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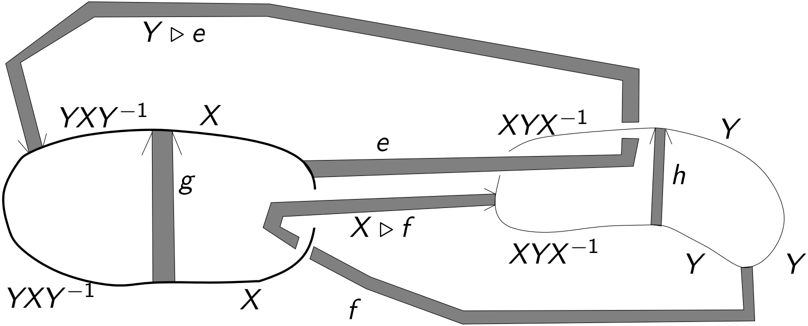

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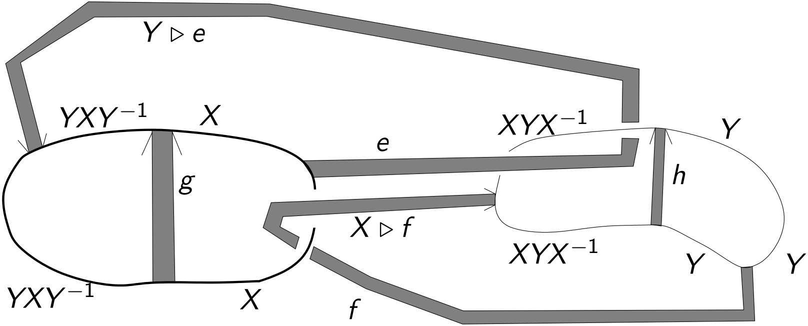

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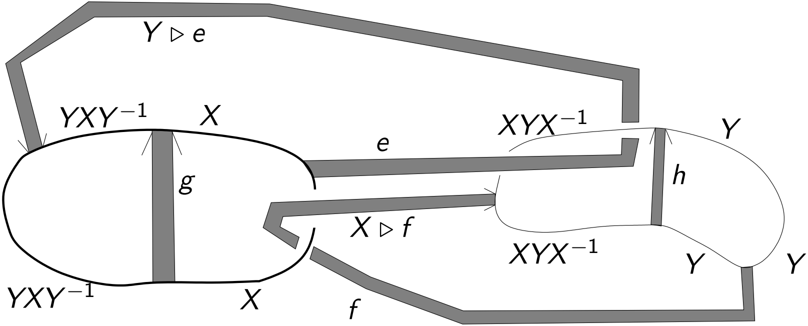

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Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e, f, g, h\} \xrightarrow[]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]\\ h\mapsto [X,Y]}} \mathcal{F}(X, Y) \;\middle|\; \substack{(Y\triangleright e)\, e^{-1}\\ (X\triangleright f^{-1})\, f\\ =1} \right\rangle$$

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$I_{\mathcal{G}}$ can distinguish Σ' from $\Sigma =$ knotted $T^2 \sqcup T^2$ above.

$\Sigma' = $ Spun Hopf Link. $M = S^4 \setminus \Sigma$

Hence

$$\Pi_2(M, M^{(1)}) = \mathcal{U}\left\langle \{e,f,g,h\} \xrightarrow[h\mapsto[X,Y]]{\substack{e\mapsto 1\\ f\mapsto 1\\ g\mapsto [Y,X]}} \mathcal{F}(X,Y) \;\middle|\; \underbrace{(Y\triangleright e)\, e^{-1}\; (X\triangleright f^{-1})\; f}_{=1} \right\rangle$$

$\pi_1(M) = \langle\{X,Y\}|[X,Y]=1\rangle$, free abelian group on X and Y.

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Let $\mathcal{G} = (\partial\colon E \to G)$ be a finite crossed module.

Recall $I_{\mathcal{G}}(M) = \frac{1}{\# E^{b_1(M^1)}} \# \hom(\Pi_2(M, M^1), \mathcal{G})$

- The invariant of knotted surfaces:

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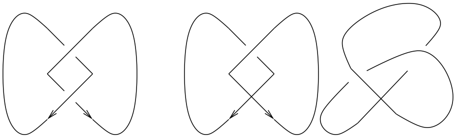

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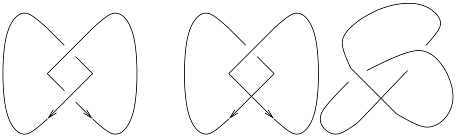

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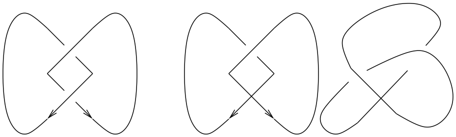

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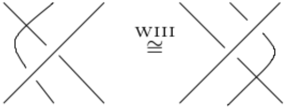

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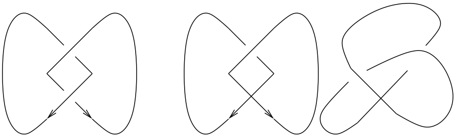

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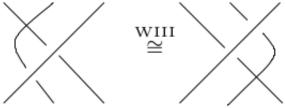

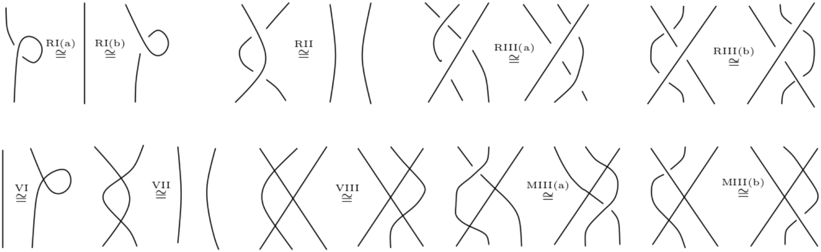

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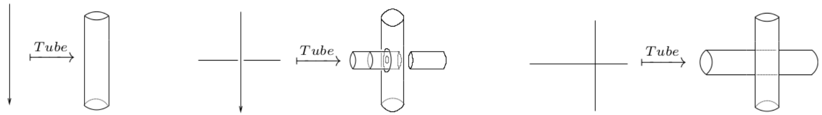

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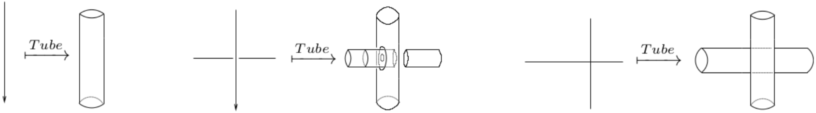

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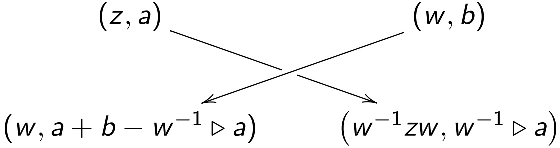

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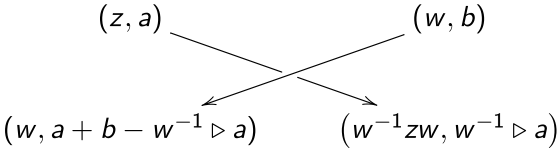

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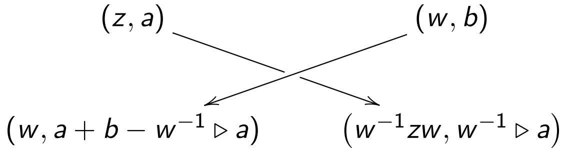

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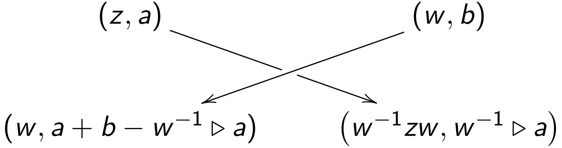

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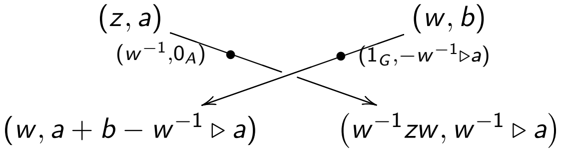

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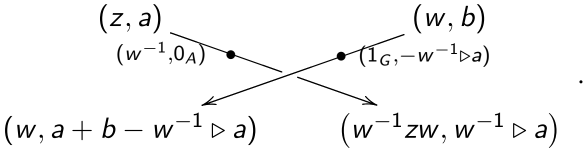

Arrows in ‘wreath product’ groupoid $\big((G \ltimes A)//(G \ltimes A)\big)^n \rtimes S_n$.

Inclusion of peripheral system information interpreted in terms of Aharonov-Bohm like effects for loop-particles moving in topological (3+1)-dimensional Higher Gauge Theory.

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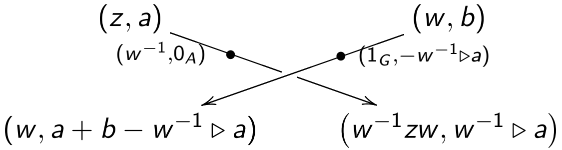

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Higher order Artin representation

Theorem

There is a representation of the Loop Braid Group LB_n *on* $\mathrm{Aut}(\mathcal{M}_n)$.

Here

$$\mathcal{M}_n = \Pi_2\Big(\vee_{i=1}^{2} (S^2 \vee S^1), \vee_{i=1}^{2} S^1 \Big)$$

Formulae are dual to those of the biquandle.

See

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THANKS!