Quinn finite total homotopy TQFT as a once-extended TQFT

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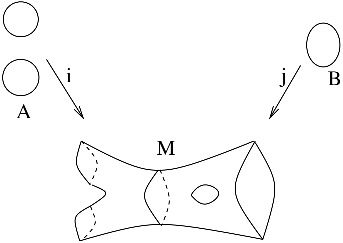
The category of manifolds and cobordisms (sketch)

Let $n \in \mathbb{Z}_0^+$. Define symmmetric monoidal category $\mathbf{Cob}^{(n,n+1)}$.

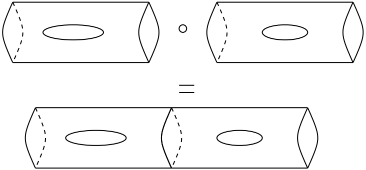
- Objects: closed smooth n-manifolds $A, B, \ldots$
- Morphisms $[M]\colon A \to B$ are equivalence classes of diagrams:

$$A \xrightarrow{i} M \xleftarrow{j} B\ .$$

Here M is a smooth $(n+1)$-manifold, and
i and j induce a diffeomorphism $\langle i, j\rangle\colon A \sqcup B \to \partial(M)$.



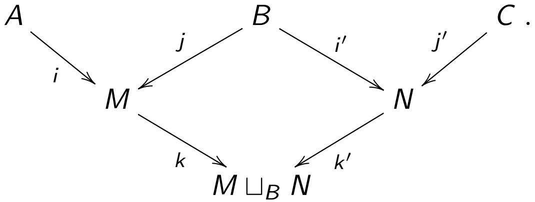
visualization.



Composition of morphisms.
Note: collars are required to construct smooth structure.

The category of manifolds and cobordisms (sketch)

More precisely, the composition of cobordisms is via pushouts:



So $([M]\colon A \to B) \bullet ([N]\colon B \to C) = ([M \sqcup_B N]\colon A \to C)$.

Note that smooth structure on $M \sqcup_B N$ is not uniquely defined.
But it is unique up to diffeomorphism.

The monoidal structure in $\mathbf{Cob}^{(n,n+1)}$ is induced from the disjoint union of manifolds / cobordisms.

Topological quantum field theories

Definition (TQFT)

Given a non-negative integer $n \in \mathbb{Z}_0^+$,
a *Topological Quantum Field Theory (TQFT)*
is a symmetric monoidal functor:

$$\mathcal{F}\colon \mathbf{Cob}^{(n,n+1)} \to \mathbf{Vect}.$$

The monoidal structure in $\mathbf{Vect}$ is given by usual tensor product.

Plan of the talk

In this talk I will:

1. Review Quinn's finite total homotopy TQFT, where $n \in \mathbb{Z}_{\geq 0}$,

   $$\mathcal{F}_{\boldsymbol{\mathcal{R}}} \colon \mathbf{Cob}^{(n,n+1)} \to \mathbf{Vect}.$$

   Here $\boldsymbol{\mathcal{R}}$ is a "homotopically finite space", a parameter.

   Cf. Frank Quinn. *Lectures on axiomatic topological quantum field theory.* In Geometry and quantum field theory. (1995)

2. Explain the combinatorial calculation of $\mathcal{F}_{\boldsymbol{\mathcal{R}}}$,
   for $\boldsymbol{\mathcal{R}}$ classifying space of a homotopy finite strict ω-groupoid (represented by a crossed complex of groupoids).

3. Explain construction / computation of once-extended $\mathcal{F}_{\boldsymbol{\mathcal{R}}}$.

Homotopy finite spaces

Definition (Homotopy finite space)

A space X is called *homotopy finite* (HF) if:

- X has only a finite number of path components.
- Given a path-component K of X, exists $n \in \mathbb{N}$ such that:
  - $\pi_i(K)$ is trivial, if $i > n$.
  - $\pi_i(K)$ finite, if $i = 1, \ldots, n$.

Equivalently, X has finitely many path-components, and finitely many non-trivial homotopy groups, all of which are finite.

Classifying spaces of groups, etc

Example

Let G be a finite group. Classifying space B_G is path-connected. Also:

- $\pi_1(B_G, *) \cong G$, and
- $\pi_i(B_G, *) = 0, \text{ if } i \geq 2.$

So B_G is a finite 1-type. So B_G is a HF space.

More generally, if G is a finite groupoid, or finite 2-group, then classifying space B_G is homotopy finite

More examples later.

The homotopy content of a homotopy finite space

Definition (Homotopy content)

If X is homotopy finite, the *homotopy content* of X is:

$$\chi^{\pi}(X) = \sum_{K \in \pi_0(X)} \frac{|\pi_2(K)|\ |\pi_4(K)|\ |\pi_6(K)| \dots}{|\pi_1(K)|\ |\pi_3(K)|\ |\pi_5(K)| \dots} \in \mathbb{Q}.$$

Here $\pi_0(X)$ is the set of path-components of X.

Example (Classifying spaces of finite groups)

If G is a finite group then $\chi^{\pi}(B_G) = 1/|G|$.

The homotopy content first appeared (I think) in:
Frank Quinn. *Lectures on axiomatic topological quantum field theory*. In Geometry and quantum field theory. (1995)

Some properties HF spaces and their homotopy content

- If X and Y are HF, then so are $X \times Y$ and $X \sqcup Y$, and:

$$\chi^\pi(X \times Y) = \chi^\pi(X) \times \chi^\pi(Y),$$

$$\chi^\pi(X \sqcup Y) = \chi^\pi(X) + \chi^\pi(Y).$$

- Let $p \colon E \to B$ be a fibration of HF spaces. Let $b \in B$.
  The fibre $F_b := p^{-1}(b)$ is HF.

  Moreover if B is path-connected then **(main powerhouse)**:

$$\chi^\pi(E) = \chi^\pi(B) \times \chi^\pi(F_b).$$

Cf. John C. Baez and James Dolan. *From finite sets to Feynman diagrams.* In Mathematics unlimited-2001 and beyond (2001)

Imma Gálvez-Carrillo, Joachim Kock, Andrew Tonks: *Homotopy linear algebra.* P ROY SOC EDINB A. (2018).

Function spaces and homotopy finite spaces

Theorem (Quinn)

Let M be a compact CW-complex, and $\boldsymbol{\mathcal{R}}$ a homotopy finite space. Then the function space, below, is homotopy finite:

$$\mathrm{TOP}(M, \boldsymbol{\mathcal{R}}) = \{f \colon M \to \boldsymbol{\mathcal{R}} \mid f \text{ is continuous}\}.$$

In particular if M is a compact smooth manifold.

Note that $\mathrm{TOP}(M, \boldsymbol{\mathcal{R}})$ is given the k-ification of the compact-open topology on the space of maps $M \to \boldsymbol{\mathcal{R}}$.

We will use the notation,

$$\mathrm{TOP}(M, \boldsymbol{\mathcal{R}}) = \boldsymbol{\mathcal{R}}^M,$$

and $\mathrm{PC}_x(X)$ denotes the path component of x in a space X.

Quinn's (finite total homotopy) TQFT

Let $\mathcal{R}$ be a HF-space. Let $s \in \mathbb{C}$.
Functor: $\mathcal{F}^{(s)}_{\mathcal{R}} \colon \mathbf{Cob^{(n,n+1)}} \to \mathbf{Vect}$.

- If A is an n-manifold then:

  $$\mathcal{F}^{(s)}_{\mathcal{R}}(A) = \mathbb{C}\big(\pi_0(\mathcal{R}^A)\big) = \mathbb{C}\{\text{homotopy classes of maps } A \to \mathcal{R}\}.$$

- Matrix elements assigned to cobordisms, $A \xrightarrow{i} M \xleftarrow{j} B$,

$$\Big\langle [f] \Big| \mathcal{F}^{(s)}_{\mathcal{R}}(M) \Big| [f'] \Big\rangle := \chi^{\pi}\left\{ H\colon M \to \mathcal{R} : \begin{array}{ccccc} A & \xrightarrow{i} & M & \xleftarrow{j} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle f'} & \\ & & \mathcal{R} & & \end{array} \text{ commutes} \right\}$$

$$\times \chi^{\pi}\big(\mathrm{PC}_f(\mathcal{R}^A)\big)^s \ \chi^{\pi}\big(\mathrm{PC}_{f'}(\mathcal{R}^B)\big)^{1-s}.$$

Proof $\mathcal{F}_{\mathcal{R}}\colon \mathbf{Cob}^{(n,n+1)} \to \mathbf{Vect}$ is a functor

Given composable cobordisms,

$$\begin{array}{ccccccc} A & & & B & & & C, \\ & \overset{i}{\searrow} & \overset{j}{\swarrow} & & \overset{i'}{\searrow} & \overset{j'}{\swarrow} & \\ & M & & & & N & \\ & & \overset{k}{\searrow} & & \overset{k'}{\swarrow} & & \\ & & & M \sqcup_B N & & & \end{array}$$

then $\mathcal{F}_{\mathcal{R}}(N\colon B \to C) \circ \mathcal{F}_{\mathcal{R}}(M\colon A \to B) = \mathcal{F}_{\mathcal{R}}(M \sqcup_B N\colon A \to C)$.

- If Σ is a submanifold of W, then restriction map, is a fibration

$$\mathrm{TOP}(W, \mathcal{R}) \to \mathrm{TOP}(\Sigma, \mathcal{R})$$
$$f \mapsto f_{|\Sigma}.$$

- If $p\colon E \to X$ is any fibration of HF spaces then

$$\chi^{\pi}(E) = \sum_{[x]\in\pi_0(X)} \chi^{\pi}(p^{-1}(x))\ \chi^{\pi}(\mathrm{PC}_x(X)).$$

Discussion: Quinn finite total homotopy TQFT

Note: Quinn TQFT $\mathcal{F}_{\boldsymbol{\mathcal{R}}}$ can be twisted by classes in $H^{n+1}(\boldsymbol{\mathcal{R}}, U(1))$.

- Let G be a finite group. Let $\boldsymbol{\mathcal{R}}$ be classifying space of G. Then $\mathcal{F}_{\boldsymbol{\mathcal{R}}}$ coincides with Dijkgraaf-Witten TQFT.

  Explicitly calculable. Related to topological gauge theory.
  Related to Kitaev quantum double model.

- Let $\mathcal{G}$ be a finite 2 group. Let $\boldsymbol{\mathcal{R}}$ be classifying space of $\mathcal{G}$. $\mathcal{F}_{\boldsymbol{\mathcal{R}}}$ coincides with (twisted) Yetter TQFT (___ / Porter).

  Explicitly calculable. Related to topological higher gauge theory.
  Related to higher Kitaev models formulated with 2-groups.

This 'computability' of Quinn generalises.

Quinn finite total homotopy TQFT is 'computable'

Theorem (Ellis)

Any connected homotopy finite space is homotopic to a space of the form $|\overline{W}(G)|$, where G is a finite simplicial group.

Graham Ellis: *Spaces with finitely many non-trivial homotopy groups all of which are finite.* Topology (1997)

Let $\mathcal{R} = |\overline{W}(G)|$, and $M\colon A \to B$ a triangulated cobordism.
Can compute $\mathcal{F}_{\mathcal{R}}(M\colon A \to B)$ using simplicial homotopy tools.

Quinn TQFT,

$$\mathcal{F}_{\mathcal{R}}\colon \mathbf{Cob}^{(n,n+1)} \to \mathbf{Vect},$$

thus is 'computable' in finite time.

The case of classifying spaces of ω-groupoids

Remainder of this talk.
We will mainly work in the case when:

$\mathcal{R}$ is the classifying space of a ω-groupoid / *crossed complex*.

E.g. $\mathcal{R}$ is the classifying space of a strict 2-group.

In this case the computation of $\mathcal{F}_{\mathcal{R}}$ is particularly simple.

Warning: Crossed complexes do not model all homotopy types. For instances, spaces modelled by crossed complexes have trivial Whitehead products.

Crossed modules of groups (as models for 2-types)

Definition (Crossed module)

A *crossed module* $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ is given by:

- A group map (i.e. a homomorphism) $\partial\colon E \to G$.
- A left action $\triangleright$ of G on E, by automorphisms,

  such that the following conditions *(Peiffer equations)* hold:

Peiffer 1 $\partial(g \triangleright e) = g\,\partial(e)\,g^{-1}, \text{ where } g \in G, e \in E;$

Peiffer 2 $\partial(e) \triangleright f = e\,f\,e^{-1}, \text{ where } e, f \in E.$

Theorem (Whitehead-MacLane)

Homotopy category of crossed modules is equivalent to homotopy category of pointed 2-types. (Pointed space with $\pi_i = 0$, if $i \geq 3$.)

Crossed complexes

A crossed complex is given by a complex of groups,

$$\mathcal{C} := \ldots \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} C_3 \xrightarrow{\partial} E \xrightarrow{\partial} G,$$

such that:

- we have a crossed module, $(\partial\colon E \to G, \triangleright)$, so G acts on E.
- all groups for C_i,for $i \geq 3$ are abelian.
- we have an action of G on all groups C_i, $i \geq 2$
- all boundary maps preserve actions,
- the action of $\partial(E) \leq G$ is trivial on all groups C_i, $i \geq 3$.

Note: Both notions, crossed modules and crossed complexes, extended to the groupoid setting.

Monoidal closed category Crs

The category Crs of crossed complexes is equivalent to the category of strict omega-groupoids (Brown–Higgins).

Also: Crs is a monoidal closed category (Brown–Higgins).

- Given $\mathcal{A}$ and $\mathcal{B}$ we can form tensor product $\mathcal{A} \otimes \mathcal{B}$.
- Given $\mathcal{A}$ and $\mathcal{B}$ we have "function space" $\mathrm{CRS}(\mathcal{A}, \mathcal{B}) = \mathcal{B}^{\mathcal{A}}$.
- Natural equivalence $\mathrm{Crs}(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \cong \mathrm{Crs}(\mathcal{A}, \mathcal{C}^{\mathcal{B}})$.

Example

Let G and H be finite groups, seen as a crossed complexes:

- $G \otimes H$ is the free product $G * H$,
- $\mathrm{CRS}(G, H)$ is the groupoid with:
  - objects maps $f\colon G \to H$,
  - morphisms $f \xrightarrow{h} f'$ are elements of H conjugating f into f'.

Fundamental crossed complexes of CW-complexes

Theorem (Brown-Higgins)

Let X be a CW-complex. Then the sequence of groupoids

$$\Pi(X) := \ldots \xrightarrow{\partial} \pi_n(X^n, X^{n-1}, X^0) \xrightarrow{\partial} \pi_{n-1}(X^{n-1}, X^{n-2}, X^0) \\ \xrightarrow{\partial} \ldots\ldots \xrightarrow{\partial} \pi_2(X^2, X^1, X^0) \xrightarrow{\partial} \pi_1(X^1, X^0) \underset{t}{\overset{s}{\rightrightarrows}} X_0.$$

is a totally free crossed complex, of groupoids.

Classifying spaces of crossed complexes

Definition (Nerve and classifying space of crossed complexes)

The nerve $\mathcal{N}\mathcal{C}$ of the crossed complex, $\mathcal{C}$, is the simplicial set $\mathcal{N}\mathcal{C}$ such that

$$(\mathcal{N}\mathcal{C})_n = \hom_{\mathrm{Crs}}\left(\Pi(\Delta(n)), \mathcal{C}\right).$$

The classifying space of $\mathcal{C}$ is $B_{\mathcal{C}} := |\mathcal{N}\mathcal{C}|$.

Theorem (Brown-Higgins)

The homotopy groups of $B_{\mathcal{C}}$ are the homology groups of $\mathcal{C}$.

So if $\mathcal{C}$ is finite then $B_{\mathcal{C}}$ is a HF space.

Calculation of Quinn's $\mathcal{F}_{\mathcal{R}}$ for $\boldsymbol{\mathcal{R}} = B_{\mathcal{C}}$.

Let $\mathcal{C}$ be a homotopically finite crossed complex
Hence classifying space $B_{\mathcal{C}}$ is a homotopically finite space.

Theorem (___ /Porter, following Brown-Higgins & Tonks)

Let A be a closed n-manifold with a triangulation. Then

$$\mathcal{F}_{B_{\mathcal{C}}}(A) \cong \mathbb{C}\pi_0(\mathcal{C}^{\Pi(A_t)}).$$

A basis of $\mathcal{F}_{B_{\mathcal{C}}}(A)$ hence consists of morphisms $f : \Pi(A_t) \to \mathcal{C}$, considered up to homotopy / pseudonatural equivalence.

Combinatorial calculation of Quinn's $\mathcal{F}_{B_{\mathcal{C}}}$

Consider a cobordism, $A \xrightarrow{\;i\;} M \xleftarrow{\;j\;} B$,

and a triangulation t of M extending triangulations A and B.

Theorem (___/Porter, following Brown-Higgins & Tonks)

Given crossed complex maps $f\colon \Pi(A_t) \to \mathcal{C}$ *and* $f'\colon \Pi(A_t) \to \mathcal{C}$

$$\left\langle [f] \middle| \mathcal{F}_{B_{\mathcal{C}}}(M) \middle| [f'] \right\rangle = \#\left\{ H\colon \Pi(M_t) \to \mathcal{C} : \begin{array}{ccccc} \Pi(A_t) & \xrightarrow{\;i\;} & \Pi(M_t) & \xleftarrow{\;j\;} & \Pi(B_t) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle H} & \swarrow{\scriptstyle f'} & \\ & & \mathcal{C} & & \end{array} \textit{ commutes} \right\}$$

$\times$*factor depending only on number of simplices of* A_t, B_t, M_t; *and* $\mathcal{C}$.

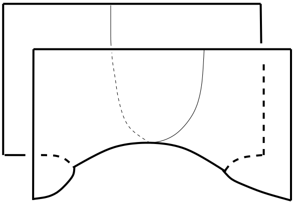
Extended cobordisms

Let $\mathbf{Cob}^{(n,n+1,n+2)}$ be the bicategory with:

- Objects n-dimensional closed smooth manifolds $A, B, \ldots$
- 1-morphisms $M\colon A \to B$ are $(n, n+1)$-cobordisms,

$$A \xrightarrow{\;i\;} M \xleftarrow{\;j\;} B \,,$$

- 2-morphisms $[K]\colon (M\colon A \to B) \implies (N\colon A \to B)$ are $(n, n+1, n+2)$-extended-cobordisms (up to diffeomorphism):



$$\begin{array}{ccccc}
A & \xrightarrow{\;i_1\;} & M & \xleftarrow{\;j_1\;} & B \\
\downarrow{\scriptstyle \iota_0^A} & & \downarrow{\scriptstyle i_N} & & \downarrow{\scriptstyle \iota_0^B} \\
A \times [0,1] & \xrightarrow{\;i_E\;} & K & \xleftarrow{\;i_W\;} & B \times [0,1] \\
\uparrow{\scriptstyle \iota_1^A} & & \uparrow{\scriptstyle i_S} & & \uparrow{\scriptstyle \iota_1^B} \\
A & \xrightarrow[\;i_2\;]{} & N & \xleftarrow[\;j_2\;]{} & B
\end{array}$$

Once-entended TQFTs

Definition

A once-extended TQFT is a symmetric monoidal bifunctor,

$$\mathcal{F}\colon \mathbf{Cob}^{(n,n+1,n+2)} \to \mathbf{Alg}.$$

Here $\mathbf{Alg}$ is some 'algebraic' symmetric monoidal bicategory.

(Once)-entended TQFTs

Two target bicategories for symmetric monoidal bifunctors

$$\mathcal{F}\colon \mathbf{Cob}^{(n,n+1,n+2)} \to \mathbf{Alg}.$$

- $\mathbf{Alg} = \mathbf{Mor}$, with:
  - objects algebras $\mathcal{A}, \mathcal{B}$,...
  - 1-morphisms $\mathcal{M}\colon \mathcal{A} \to \mathcal{B}$ being $(\mathcal{A}, \mathcal{B})$-bimodules $\mathcal{M}$,
  - Composition $\mathcal{A} \xrightarrow{\mathcal{M}} \mathcal{B} \xrightarrow{\mathcal{N}} \mathcal{C}$ given $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M} \otimes_{\mathcal{B}} \mathcal{N}$,
  - 2-morphisms $(\mathcal{A} \xrightarrow{\mathcal{M}} \mathcal{B}) \implies (\mathcal{A} \xrightarrow{\mathcal{M}'} \mathcal{B})$ are bimodule maps.
- $\mathbf{Alg} = \mathbf{Prof}$, with:
  - objects homotopy finite groupoids $\mathcal{G}$, $\mathcal{H}$, ...
  - 1-morphisms $\mathcal{G} \to \mathcal{H}$ are functors $\mathcal{G}^{\mathrm{op}} \times \mathcal{H} \to \mathbf{Vect}$,
  - composition is usual profunctor composition (via coends),
  - 2-morphisms are natural transformations of functors.

Homotopical powerhouse for once-extended Quinn TQFT

Let **Top**$/\sim$ be category of spaces and homotopy classes of maps.

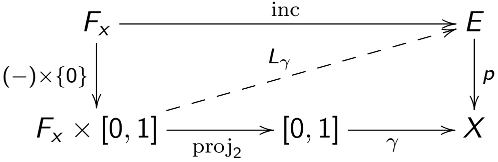
Theorem (Classical)

Let $p\colon E \to X$ be a fibration. We have a functor

$$\mathrm{Hol}\colon \pi_1(X, X) \to \textbf{Top}/\sim .$$

It sends:

- $x \in X$ *to* $F_x := p^{-1}(x)$,
- *a morphism* $[\gamma]\colon x \to y$ *to:*



So $\mathrm{Hol}(\gamma)$ *is homotopy class of* $L_\gamma(-,1)\colon F_x \to F_y$.

Main theorem: once-extended TQFT

Let $\mathcal{R}$ be a HF space. We have a once-extended TQFT

$$\widehat{\mathcal{Q}}_{\mathcal{R}} \colon \mathbf{Cob}^{(n,n+1,n+2)} \to \mathbf{Prof}.$$

- If A is an n manifold then $\widehat{\mathcal{Q}}_{\mathcal{R}}(A) = \pi_1(\mathcal{R}^A, \mathcal{R}^A)$.
- A cobordism, $M \colon A \to B = \quad A \xrightarrow{i} M \xleftarrow{j} B$,

  gives rise to 'path-space fibration':

  $$\langle i^*, j^* \rangle \colon \mathcal{R}^M \to \mathcal{R}^A \times \mathcal{R}^B.$$

  The profunctor, $\widehat{\mathcal{Q}}_{\mathcal{R}}(M : A \to B) \colon \widehat{\mathcal{Q}}_{\mathcal{R}}(A) \not\to \widehat{\mathcal{Q}}_{\mathcal{R}}(B)$, is:

  $$\widehat{\mathcal{Q}}_{\mathcal{R}}(A)^{\mathrm{op}} \times \widehat{\mathcal{Q}}_{\mathcal{R}}(B) \xrightarrow{Hol} \mathbf{Top}/\sim \xrightarrow{\pi_0} \mathbf{Set} \xrightarrow{\mathrm{Lin}} \mathbf{Vect}.$$

- As per Quinn TQFT from here: non trivial it works this way.

Decorated manifolds

Let $\mathcal{R}$ be a homotopy finite space.

Note that in general $\pi_1(\mathcal{R}^A, \mathcal{R}^A)$ is uncountable:
we have one object for each function $f\colon A \to \mathcal{R}$.

Let us 'reduce' the size of the target groupoids.

Definition (Decorated manifold)

An $\mathcal{R}$-decorated manifold, $\mathbf{A} = (A, \overline{x}_A)$, is a manifold, A, and a finite subset $\overline{x}_A \subset \mathcal{R}^A$, intersecting each path-component of $\mathcal{R}^A$.

We have bicategory, $\overline{\mathbf{Cob}}^{(n,n+1,n+2)}$, of $\mathcal{R}$-decorated manifolds, and (undecorated) cobordisms and extended cobordisms.

Finitary once-extended version of Quinn TQFT

Let $\boldsymbol{\mathcal{R}}$ be a homotopy finite space, n a non-negative integer.

Theorem (Finitary once-extended Quinn TQFT)

We have bifunctor:

$$\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}} \colon \overline{\mathbf{Cob}}^{(n,n+1,n+2)} \to \mathbf{Prof},$$

sending $\mathbf{A} = (A, \overline{x}_A)$ *to* $\pi_1(\boldsymbol{\mathcal{R}}^A, \overline{x}_A)$*.*

Note 1: The groupoid $\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}(\mathbf{A}, \overline{x}_A)$ is finite.

Note 2: Let A be an n-manifold.
If $\overline{x}_A$ and $\overline{y}_A$ are different decorations of A then

$$\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}\Big((A, \overline{x}_A) \xrightarrow{A \times I} (A, \overline{y}_A)\Big)$$

gives a canonical profunctor $\overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}(A, \overline{x}_A) \not\to \overline{\mathcal{Q}}_{\boldsymbol{\mathcal{R}}}(A, \overline{y}_A)$.

Morita valued extended version of Quinn TQFT

Let $\mathcal{R}$ be a homotopy finite space and $n \in \mathbb{Z}_{\geq 0}$.

Theorem (Morita valued once-extended Quinn TQFT)

The symmetric monoidal bifunctor:

$$\overline{\mathcal{Q}}_{\mathcal{R}} \colon \overline{\mathbf{Cob}}^{\boldsymbol{(n,n+1,n+2)}} \to \mathbf{Prof},$$

sending $\mathbf{A} = (A, \overline{x}_A)$ *to* $\pi_1(\mathcal{R}^A, \overline{x}_A)$*,*

"linearises" to a bifunctor, denoted:

$$\overline{\mathcal{Q}}^{\mathbf{Mor}}_{\mathcal{R}} \colon \overline{\mathbf{Cob}}^{\boldsymbol{(n,n+1,n+2)}} \to \mathbf{Mor},$$

sending $\mathbf{A} = (A, \overline{x}_A)$ *to groupoid algebra* $\mathbb{C}(\pi_1(\mathcal{R}^A, \overline{x}_A))$.

The case of crossed complexes / strict omega-groupoids

Suppose that $\mathcal{R} = B_{\mathcal{C}}$, where $\mathcal{C}$ is a finite crossed complex.

Theorem

If A has a triangulation, t, then A is naturally decorated. Moreover:

$$\overline{\mathcal{Q}}_{B_{\mathcal{C}}}(A_t) \cong \pi_1\big(\mathcal{C}^{\Pi(A_t)}\big),$$

$$\overline{\mathcal{Q}}^{\mathbf{Mor}}_{B_{\mathcal{C}}}(A_t) \cong \mathbb{C}\Big(\pi_1\big(\mathcal{C}^{\Pi(A_t)}\big)\Big).$$

Note: $\pi_1\big(\mathcal{C}^{\Pi(A_t)}\big)$ *is groupoid of morphisms* $\Pi(A_t) \to \mathcal{C}$*, and (2-fold homotopy classes of) homotopies between them.*

Hence once-extended TQFTs, $\overline{\mathcal{Q}}_{B_{\mathcal{C}}}$ & $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{B_{\mathcal{C}}}$, can be computed.

The case of crossed complexes / strict omega-groupoids

Let $\{A^i\}_{i\in I}$ be set, containing at least one representative of each diffeomorphism class of closed connected n-manifolds.

Choose a triangulation t_i of each $A^{(i)}$.

Theorem (——, Porter)

Let $n \in \mathbb{Z}_{\geq 0}$. There exist once-extended TQFTs,

$$\overline{\mathcal{Q}}_{B_{\mathcal{C}}} \colon \mathbf{Cob}^{(n,n+1,n+2)} \to \mathbf{Prof},$$

$$\overline{\mathcal{Q}}^{\mathbf{Mor}}_{B_{\mathcal{C}}} \colon \mathbf{Cob}^{(n,n+1,n+2)} \to \mathbf{Mor},$$

such that, for each i:

$$\overline{\mathcal{Q}}_{B_{\mathcal{C}}}(A^{(i)}) = \pi_1\Big(\mathcal{C}^{\Pi(A^{(i)}_{t_i})}\Big), \qquad \overline{\mathcal{Q}}^{\mathbf{Mor}}_{B_{\mathcal{C}}}(A^{(i)}) = \mathbb{C}\pi_1\Big(\mathcal{C}^{\Pi(A^{(i)}_{t_i})}\Big).$$

Bullivant: canonical Morita equivalences if triangulations change.

Some computations

Simplest case. Let G be a finite group.

- $n = 0$, then $\overline{\mathcal{Q}}_{B_G}(\cdot) = G$, and hence $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{B_G}(.) = \mathbb{C}(G)$,
- $n = 1$, then $\overline{\mathcal{Q}}_{B_G}(S^1)$=conjugation groupoid of G, so

$$\overline{\mathcal{Q}}^{\mathbf{Mor}}_{B_G}(S^1) \text{ is the quantum double of } \mathbb{C}(G).$$

  New proof that there exists a (1,2,3)-extended TQFT sending S^1 to the quantum double of G.
  Cf. Jeffrey Morton. *Cohomological twisting of 2-linearization and extended TQFT.* J. Homotopy Relat. Struct. (2015).

- $n = 2$, then $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{B_G}(S^2) = \mathbb{C}$.
- $n = 2$, then $\overline{\mathcal{Q}}^{\mathbf{Mor}}_{B_G}(S^1 \times S^1)$ is groupoid algebra of action groupoid:

$$\{(a, b) \in G \times G \mid ab = ba\}//G.$$

Some computations

Example

Let $\mathcal{G} = (\partial\colon E \to G, \triangleright)$ be a finite crossed module.

$n = 1$: $\overline{\mathcal{Q}}_{B_{\mathcal{G}}}(S^1)$ has:

- objects: $g \in G$.
- morphisms equivalence classes of arrows like:

$$g \xrightarrow{[(h,e)]} \partial(e)hgh^{-1}, \quad g, h \in G, \quad e \in E.$$

$=$ groupoid of $\mathcal{G}$-2-connections on S^1, with morphisms (2-gauge) equivalence classes of gauge transformation.

Cf. Alex Bullivant and Clement Delcamp. *Tube algebras, excitations statistics and compactification in gauge models of topological phases.* JHEP (2019)

Alex Bullivant and Clement Delcamp. *Excitations in strict 2-group higher gauge models of topological phases.* JHEP (2020).

Thanks!

Main References:

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- *A categorification of Quinn's finite total homotopy TQFT with application to TQFTs and once-extended TQFTs derived from strict omega-groupoids.* arXiv:2301.02491 [math.CT].

Some technical references to results used in the talk.

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